

4. Discrete Wavelet Transform

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3.Continuance Wavelet Transform

3.1 Introducing Wavelet Transform

3.1.1 Continuance Wavelet Transform (DWT)

3.1.2 Discrete Wavelet Transform (DWT)

3.2 Selection of MW for the CWT

3.2.1 Condition of MW selection

3.2.2 Example of flow turbulence analysis

3.3 Fast Algorithm for the CWT

3.3.1 Fast Algorithm in frequency domain

3.3.2 Improvement Fast Algorithm

3.3.3 Example of EEG analysis

3.4 Constructing new RMW

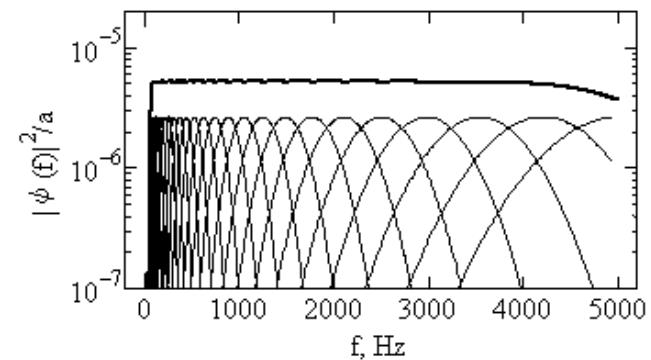
Wavelet Instantaneous Correlation (WIC)

Important point of CWT

For Wavelet Transform

Problems:

- Redundancy bases
- How do select Mother wavelet?
- There isn't a common fast algorithm for calculation.



(b) Basis of CWT for 6 octave, 4 divided

Selection of MW for the CWT

- MW types: Complex type, real type
- For different MW, analysis result is different

Fast Algorithm for the CWT

Fast Algorithm in frequency domain

Constructing new RMW

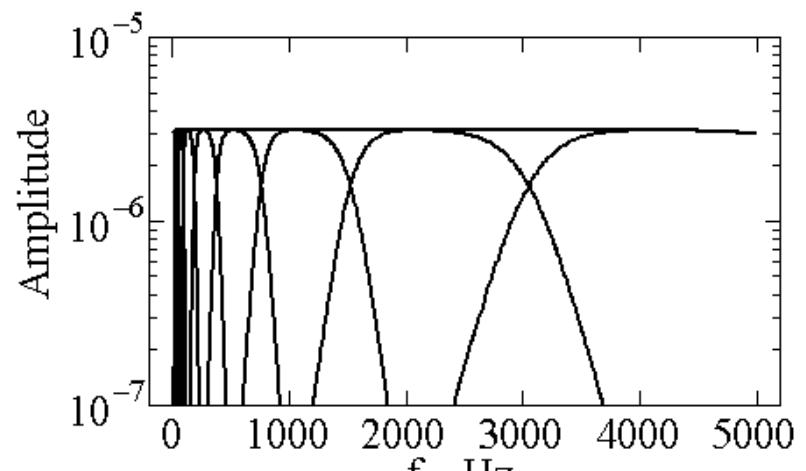
It is a new approach for abnormal signal detection

4.1 Discrete Wavelet Transform (DWT)

1) Definition of the DWT

$$d_k^j = \int_{-\infty}^{\infty} f(t) \bar{\psi}_{j,k}(t) dt$$

$$\psi_{j,k}(t) = 2^{1/2} \psi(2^j t - k)$$



It is octave analysis so bases are not overlap
(Non-redundancy)

2)Fast algorithm Based on MRA (by Mallat)

$$d_k^{j-1} = \sum_k a_{l-2k} c_k^j$$

$$c_k^{j-1} = \sum_k b_{l-2k} c_k^j$$

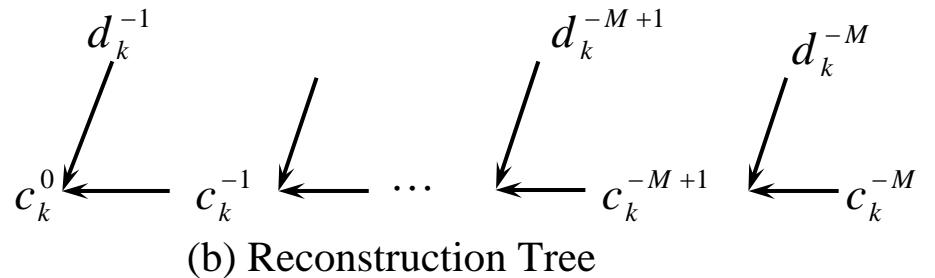
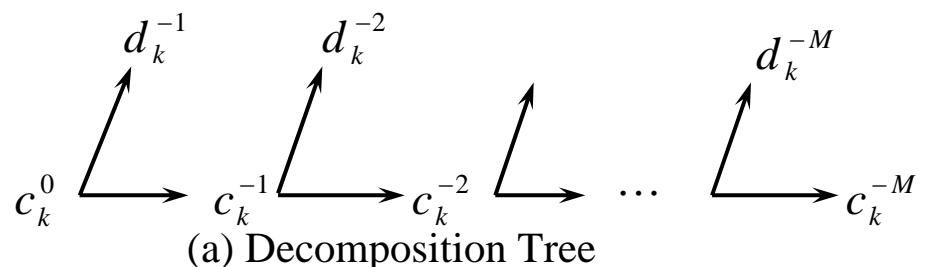


Fig.2 Tree algorithm of MRA

3) Example of the DWT

$$d_k^j = \int_{-\infty}^{\infty} f(t) \bar{\psi}_{j,k}(t) dt$$

$$\psi_{j,k}(t) = 2^{-j/2} \psi(2^j t - k)$$

j : Level

k : Time

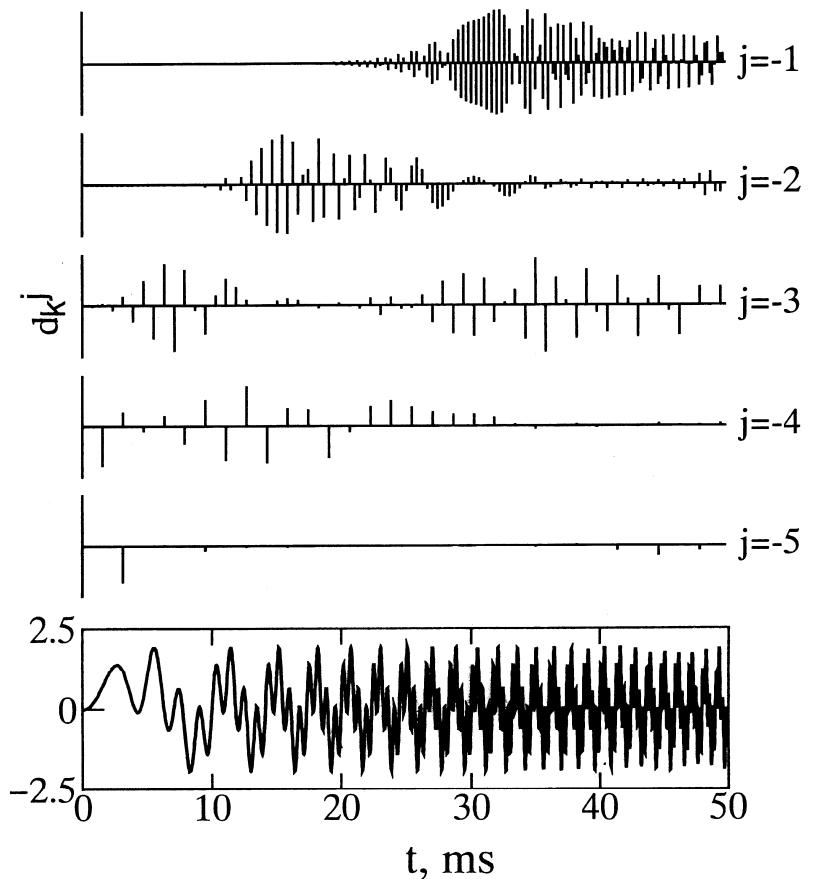


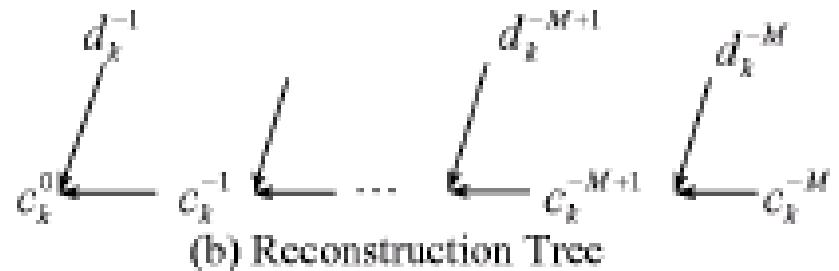
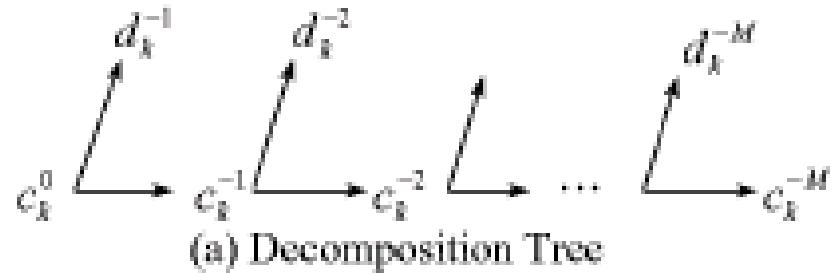
Fig.3 Discrete Wavelet Transform of Model Signal

4.2 Features of the DWT

- ★ Good compression of signal energy.
- Perfect reconstruction with short support filters.
- No redundancy. Very low computation order- N only.

But

- ★ Severe shift dependence.
- Poor directional selectivity
in 2-D,3-D etc.



The DWT is normally implemented with a tree of high pass and low pass filters, which proposed by Mallat (fast algorithm).

1)Severe shift dependence

(MW is Daubichies 8)

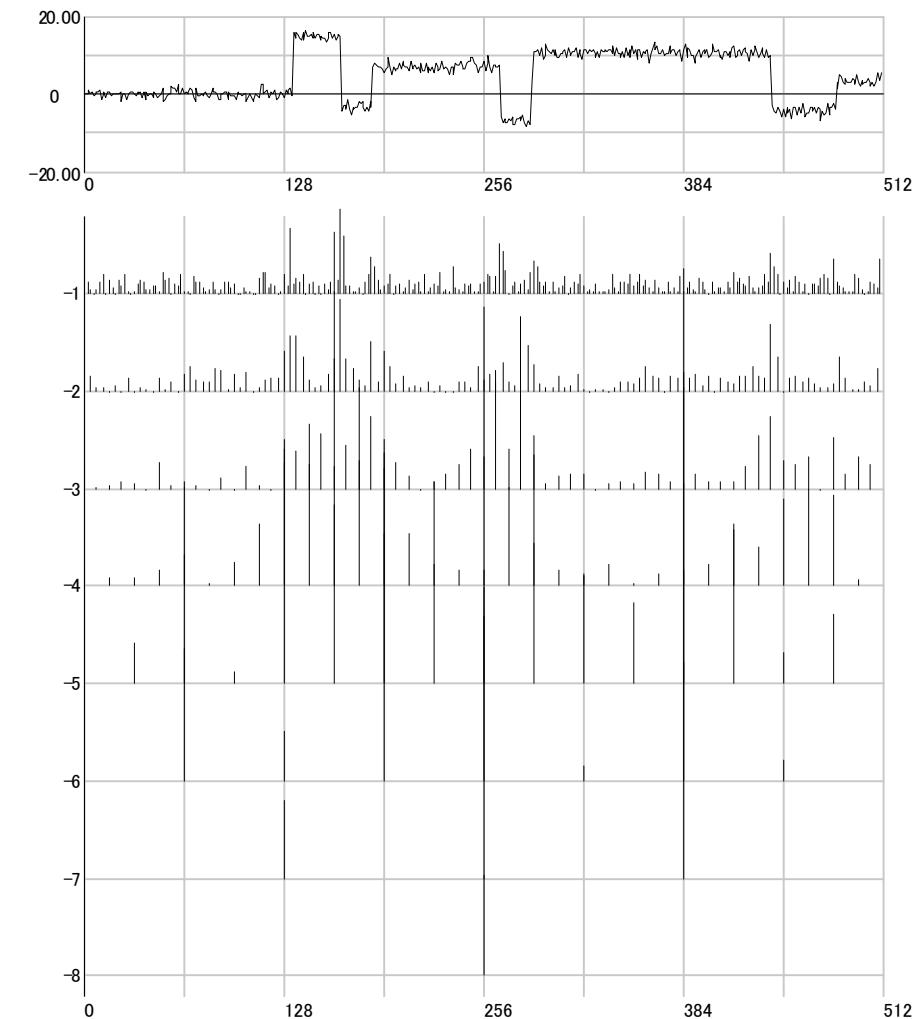


Fig.5 Original signal

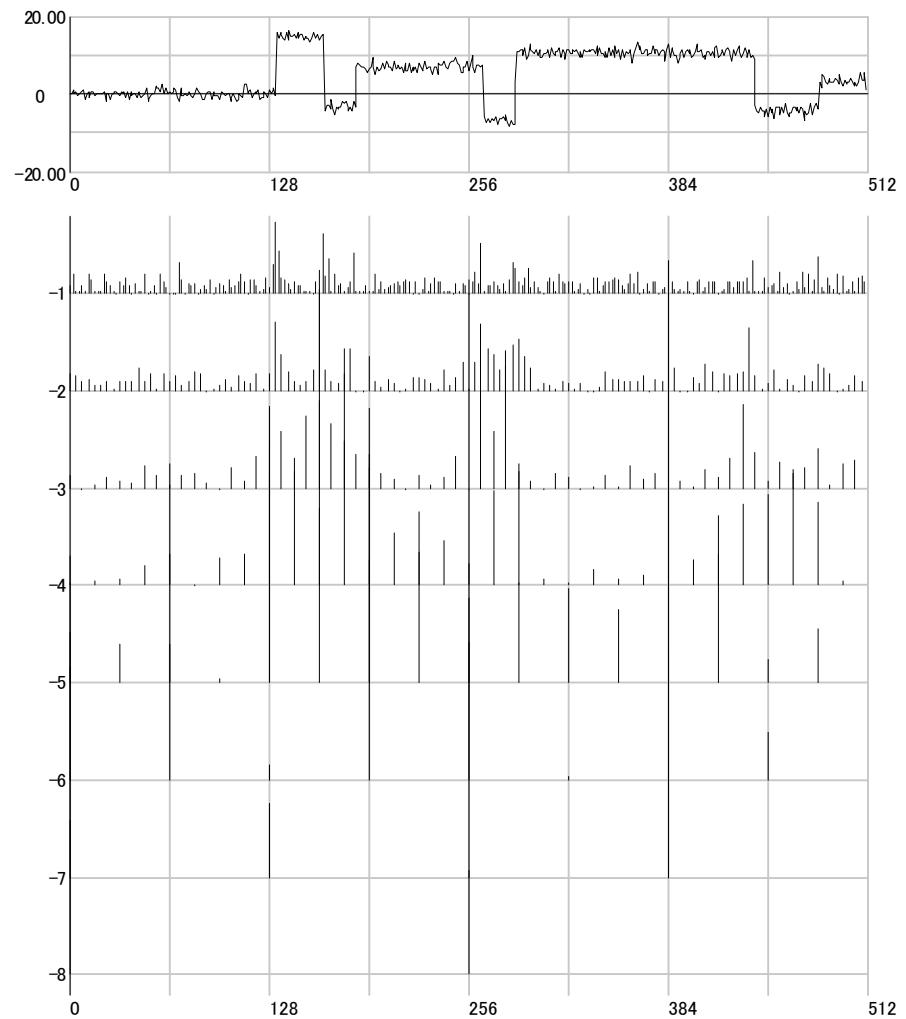


Fig.6 Signal with shift of one sample

2)Poor direction selectivity



(a) Circle Image (256×256)

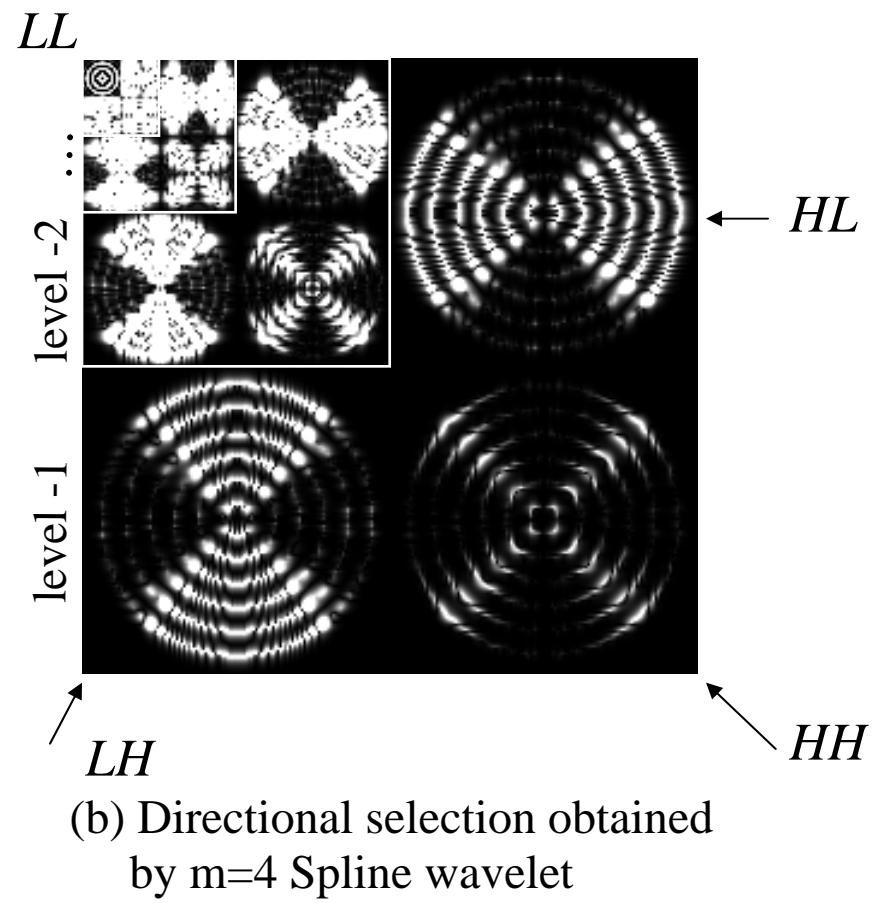


Fig.3 Wavelet transform of circle Image by DWT

Reason

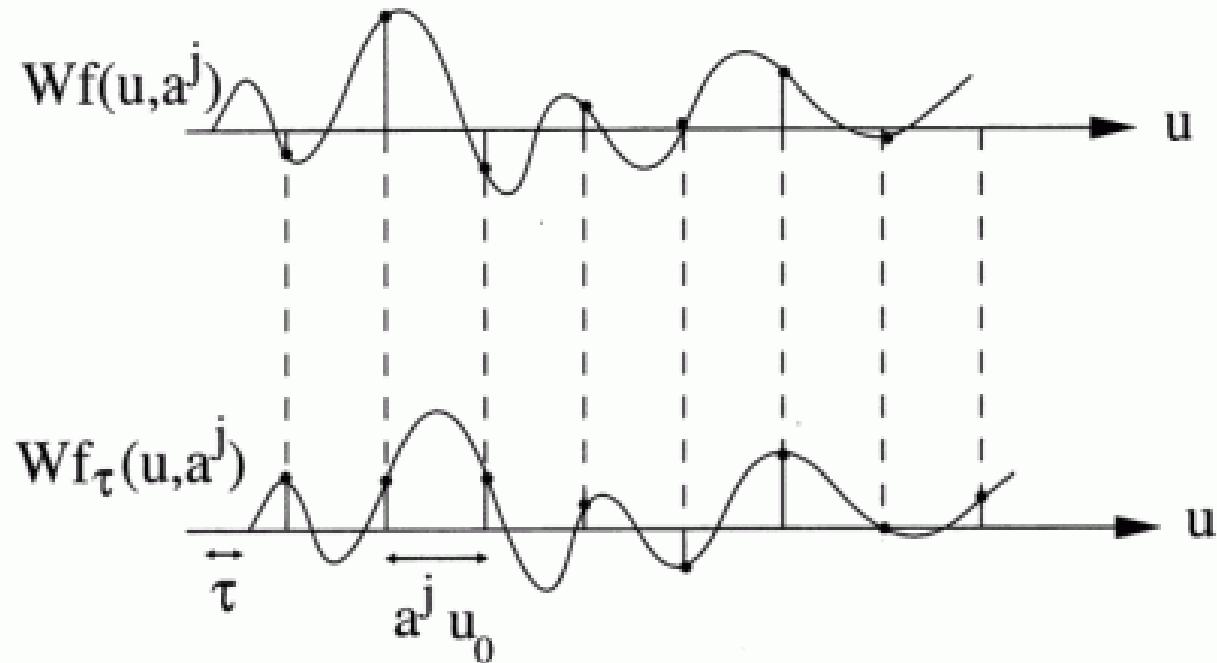
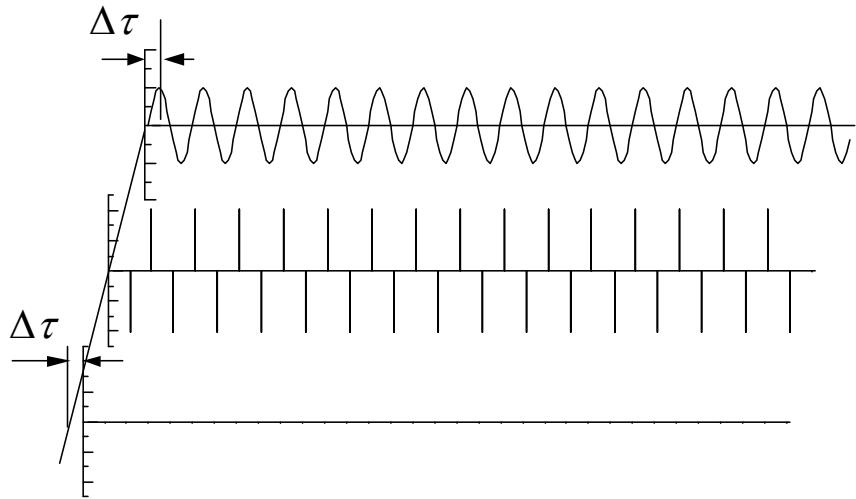


FIGURE 5.3 If $f_\tau(t) = f(t - \tau)$ then $Wf_\tau(u, a^j) = Wf(u - \tau, a^j)$. Uniformly sampling $Wf_\tau(u, a^j)$ and $Wf(u, a^j)$ at $u = na^j u_0$ may yield very different values if $\tau \neq ku_0 a^j$.

* S.Mallat, a Wavelet tour of signal processing, Academic Press (1999), p.148.

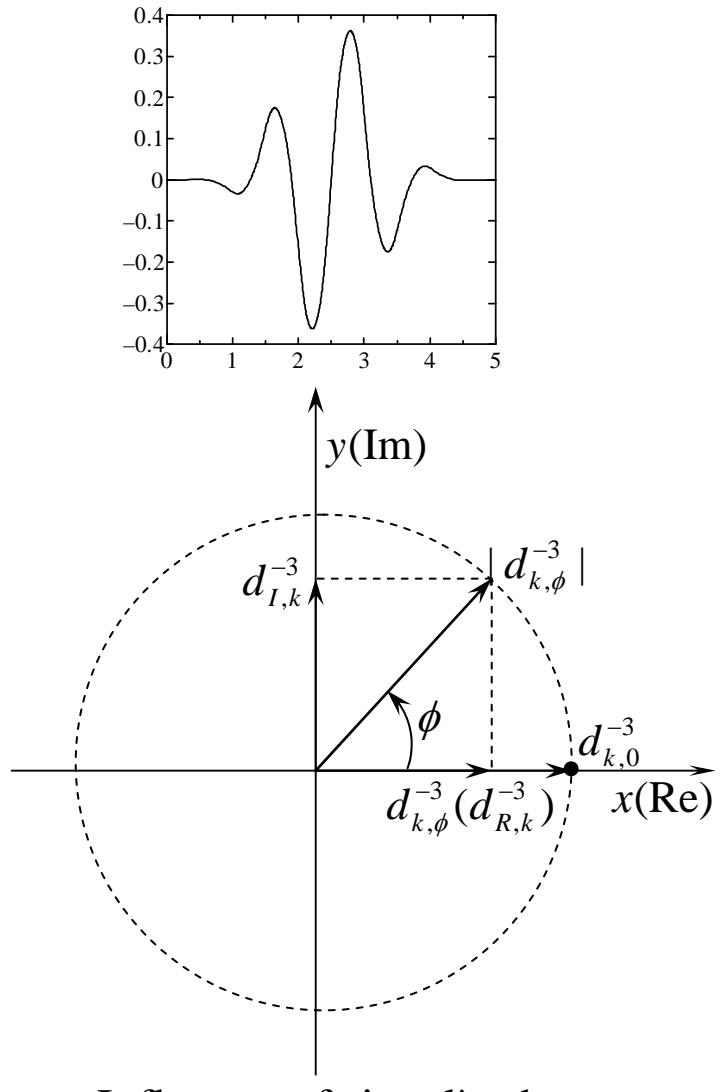
Phase



An example of wavelet coefficients in level -3 obtained by the m=4 Spline wavelet, where signal is $f(k) = \sin(\pi k / 8)$

$$|d_{k,\phi}^{-3}| = \sqrt{(d_{I,k}^{-3})^2 + (d_{R,k}^{-3})^2}$$

$|d_{k,\phi}^{-3}|$ becomes translation invariance



Influence of signal's phase

4) Methods for improving translation invariance

Some researches for improving translation invariance:

1)Using complex filter banks, such as complex Daubechies filter banks for traditional DWT.

Features: using traditional DWT

Drawback: it isn't perfect reconstruction.

2)Carrying out Hilbert transform to obtain real and imaginary components of original data, and analyzing it using traditional DWT

Features: using same DWT twice

Drawback: the information of imaginary components near Nyquist frequency have been lost.

3) Q-shift Dual Tree Complex Wavelet Transform

Q-SHIFT DUAL TREE COMPLEX WAVELET TRANSFORM IN 1-D

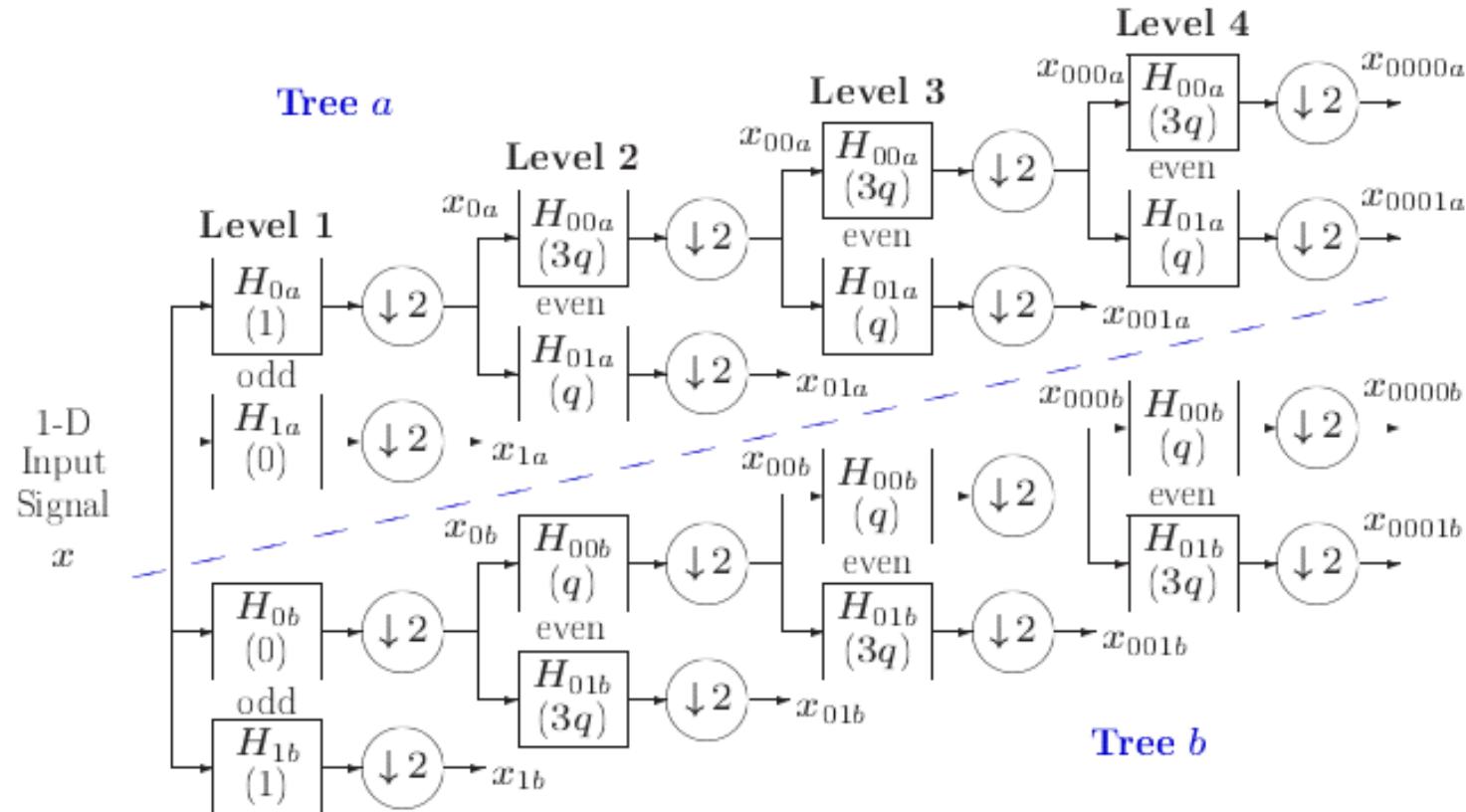


Figure 2: Dual tree of real filters for the Q-shift CWT, giving real and imaginary parts of complex coefficients from tree **a** and tree **b** respectively. Figures in brackets indicate the approximate delay for each filter, where $q = \frac{1}{4}$ sample period.

Drawback: designing filter with $q=1/4$ sample period is not easy

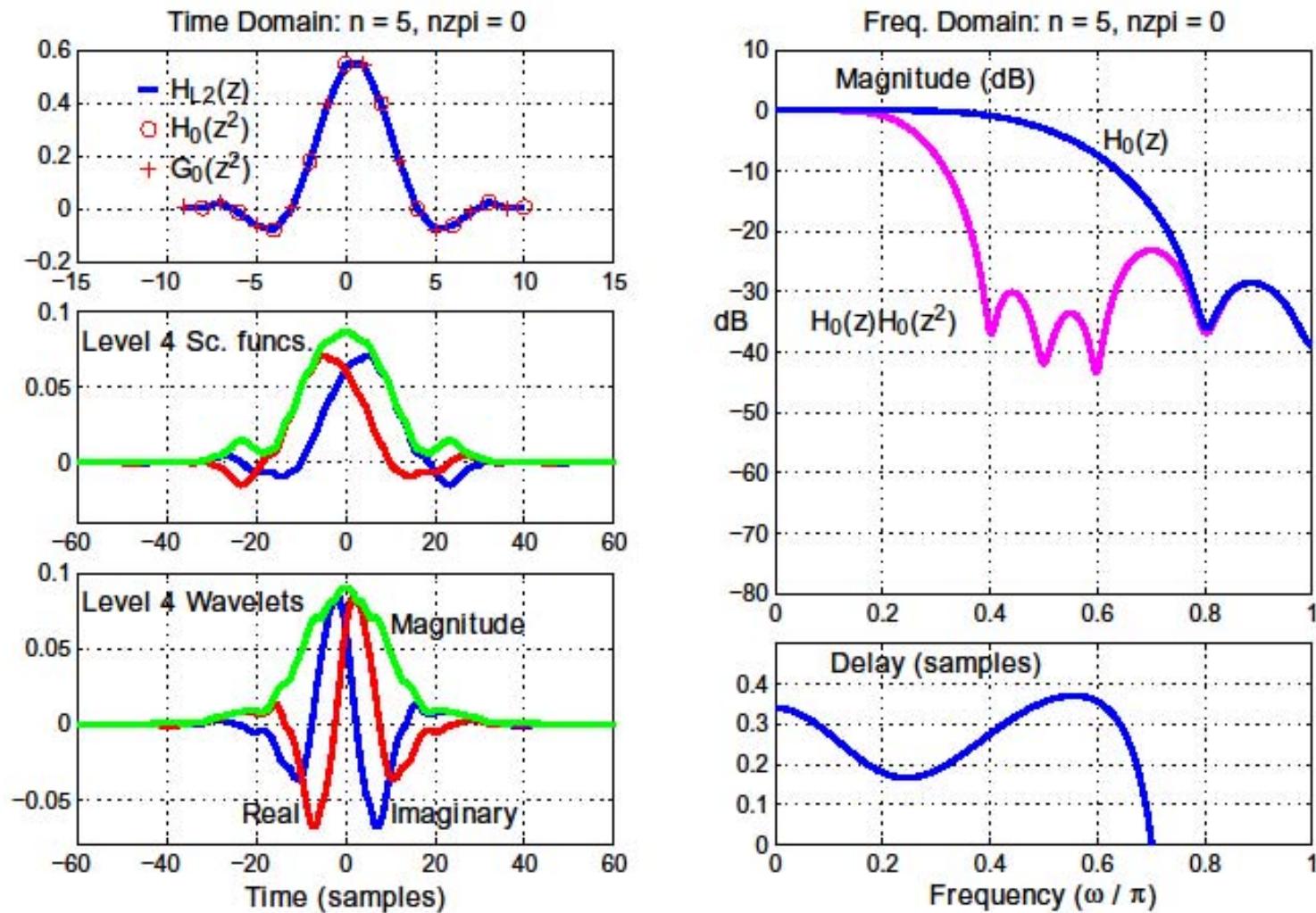
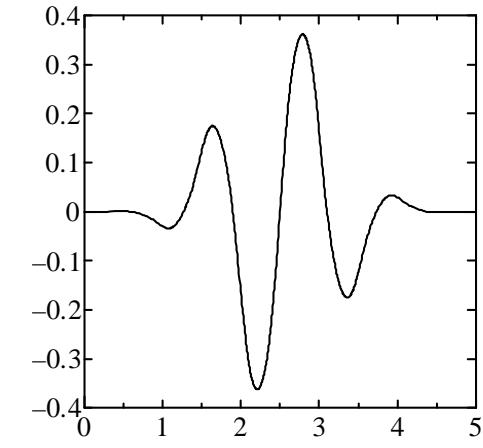
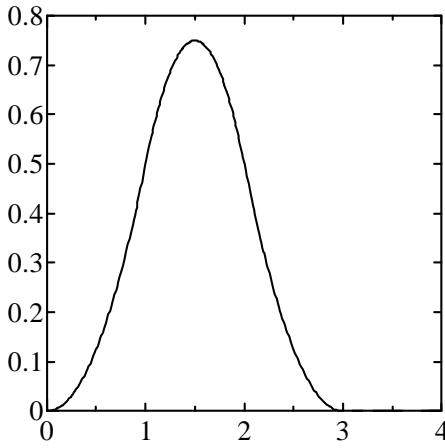


Fig. 6: Q-shift filters for $n = 5$ (10 filter taps) and no predefined zeros.

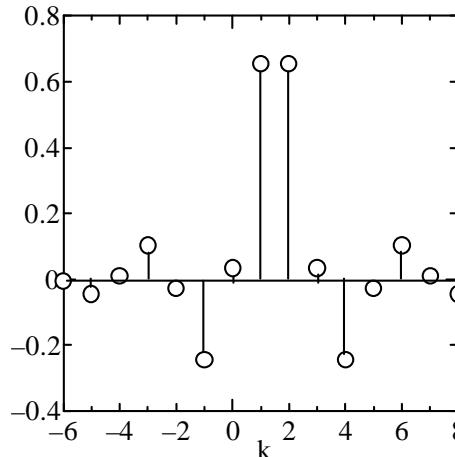
What is half sample delay?



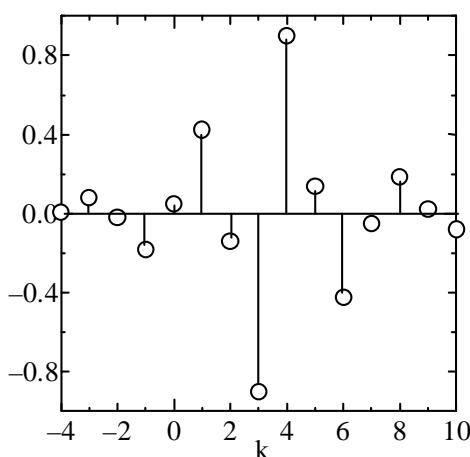
(a)Wavelet



(b)Scaling function

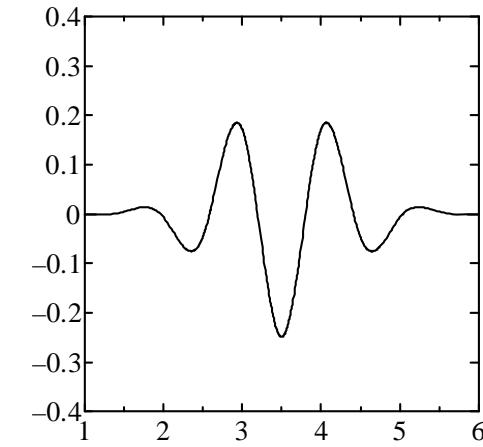


(c) a_k

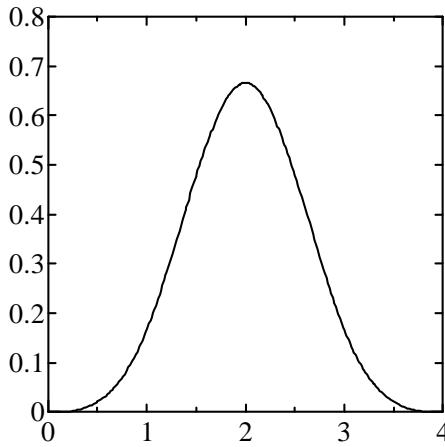


(d) b_k

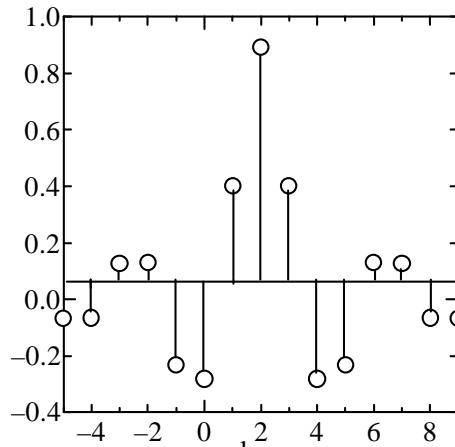
(a) $m=3$ Spline Wavelet



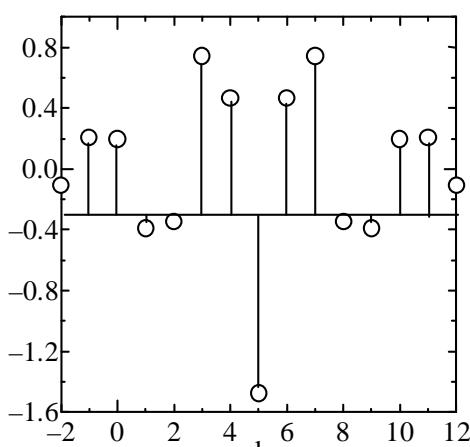
(a)Wavelet



(b)Scaling function



a_k



(d) b_k

(b) $m=4$ Spline waveley

There is a half sample delay problem between two wavelet

Two problem of DT-DWT:

- 1)How do design filter for Dual tree WT?**
- 2)How do create a half sample delay between two tree?**

4.3 New Design methods for CDWT

Translation invariance condition:

We proved that translation invariance of the Complex Discrete Wavelet Transform (CDWT) can be achieved in the following translation invariance condition:

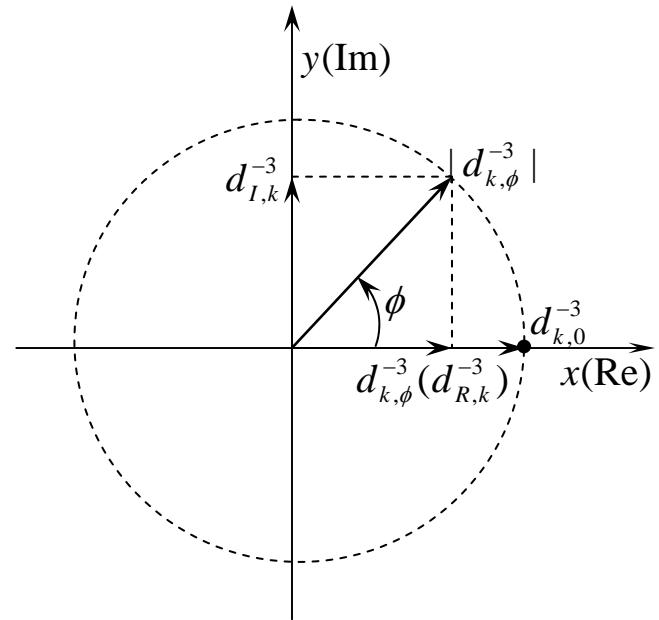
In the *scaling functions* that corresponded to the complex wavelet of Hilbert transformation pair, their shapes of the real component and the imaginary component are completely same and positions of the real component and the imaginary component are different from 1/2 sample.

4.3.1 Requirement condition (1)

Real and Imaginary components of the Complex Mother Wavelet must be a Hilbert pair

$$P^I(\omega) = P^R(\omega)e^{-i\pi/2}, \quad -\pi < \omega < \pi$$

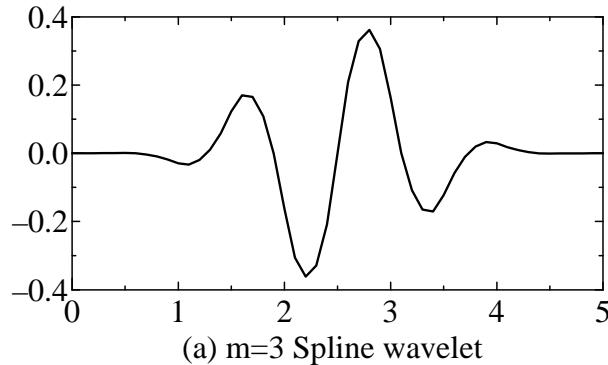
Where, $P^I(\omega), P^R(\omega)$ are frequency responses of the two scale sequences $\{p^R(\omega)\}$ and $\{p^I(\omega)\}$ respectively.



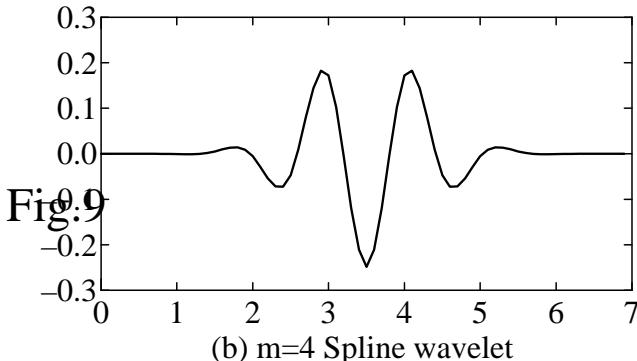
Notice: For orthogonal wavelet, all the necessary sequences (wavelet and scaling function) can be calculated from two scale sequences

1) Simple design method by using Spline wavelets

Designing translation invariant Real-Imaginary Spline Wavelet
(Bi-orthogonal RI-B-Spline wavelet)



(a) $m=3$ Spline wavelet



(b) $m=4$ Spline wavelet

Fig.9

Examples of Spline wavelet

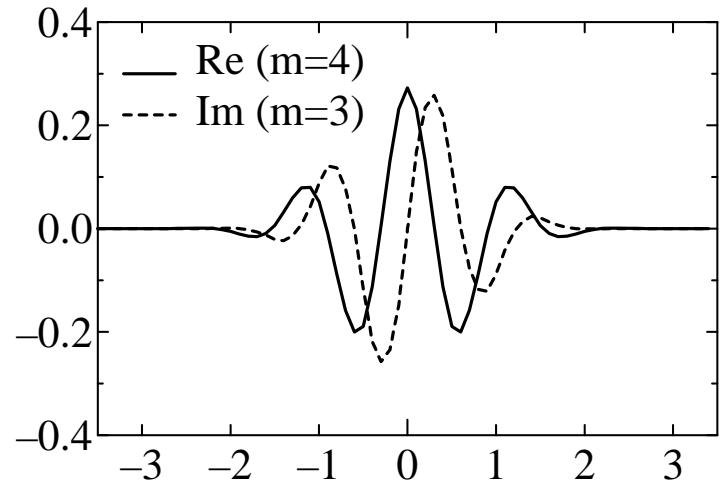
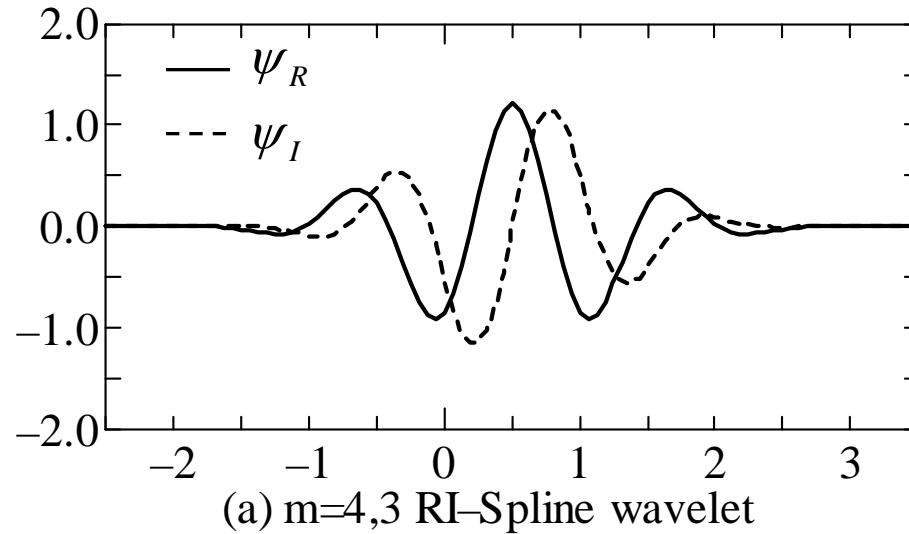


Fig.10 $m=3,4$ RI-Spline wavelet

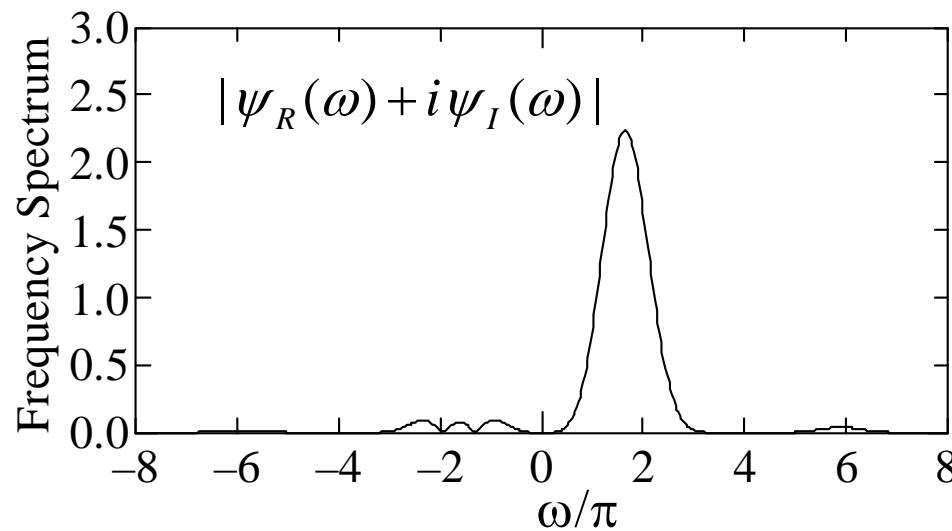
$$\psi(t) = \frac{1}{\sqrt{2}} [\psi_{m_e}(t) + i \psi_{m_o}(t)]$$

Necessary condition of the complex wavelet:

$$P_{\omega}^i(\omega) = P_{\omega}^r(\omega) e^{-i\frac{\omega}{2}}, \quad -\pi < \omega < \pi$$



(a) $m=4,3$ RI-Spline wavelet



(b) Fourier transform of $m=4,3$ RI-Spline wavelet

2) A novel design method for the Orthogonal wavelets

In order to satisfy following condition:

$$P^I(\omega) = P^R(\omega)e^{-i\pi/2}, \quad -\pi < \omega < \pi$$

1) Assuming that:

- (a) The two scaling sequence p_k^R is known
- (b) Frequency response of sequence h_n is $H(\omega) = e^{-i\omega/2}$

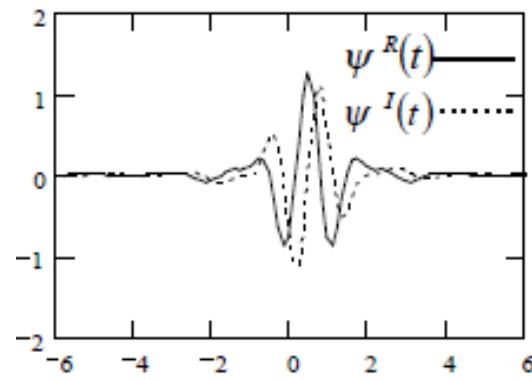
2) Calculating h_n by inverse FFT of $H(\omega)$

$$h_n = \frac{\sin(n - 1/2)\pi}{(n - 1/2)\pi}$$

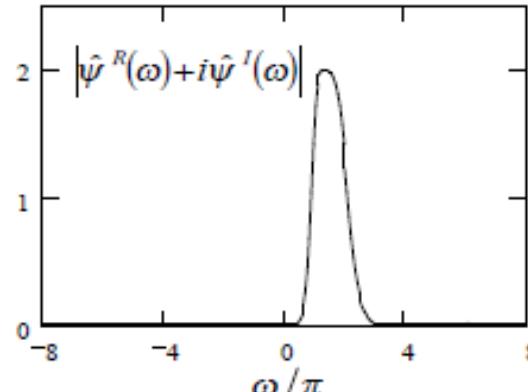
3) Calculating p_k^I using following equation:

$$p_n^I = \sum_k p_k^R h_{n-k}$$

Example f RI-Wavelet:

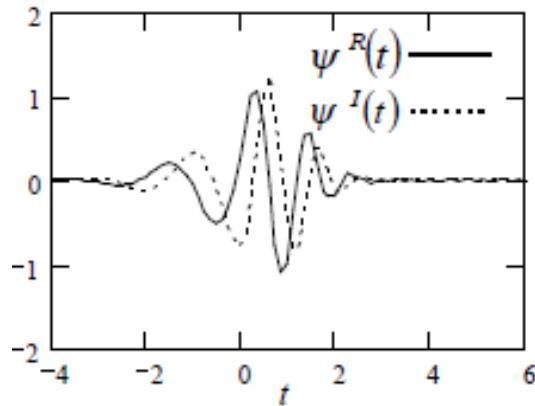


Complex MW

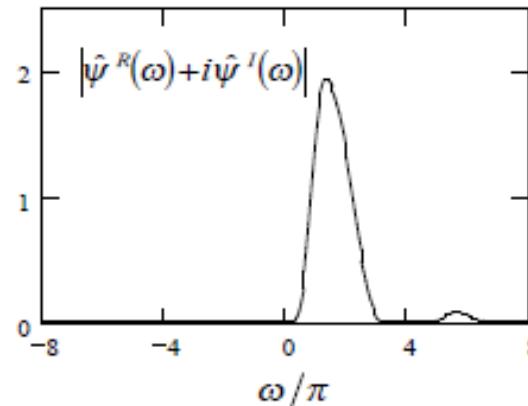


Power spectrum

(a) Example of Orthonormal RI-O-Spline wavelet



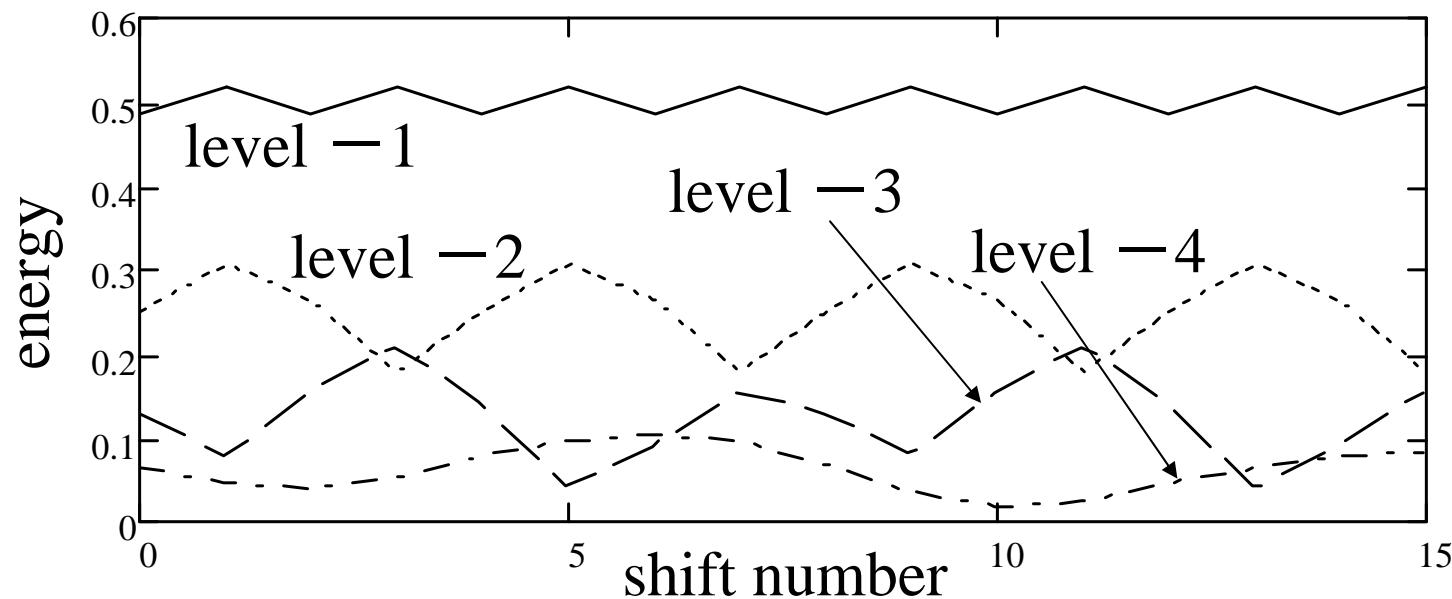
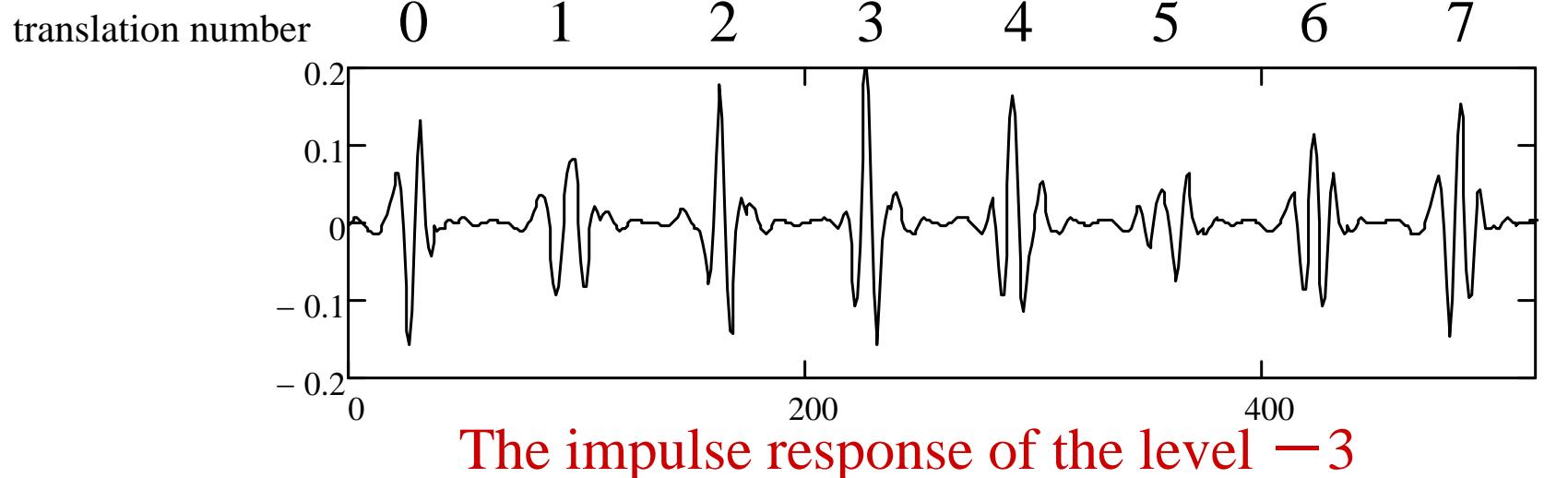
Complex MW



Power spectrum

(b) Example of Orthonormal RI-Daubechise 6 wavelet

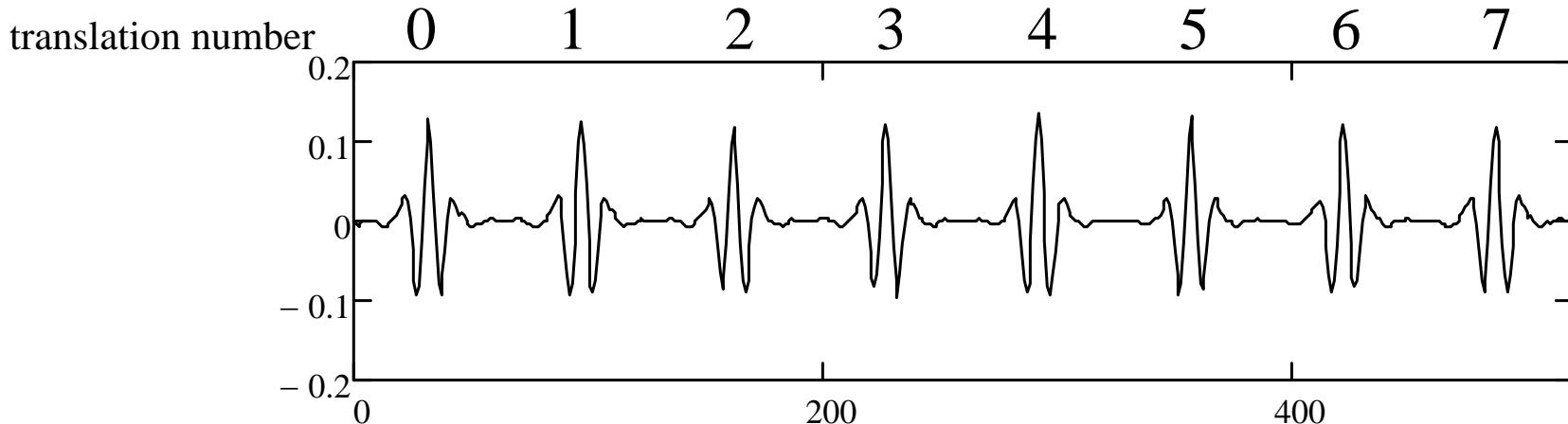
The impulse response by the Daubechies 6



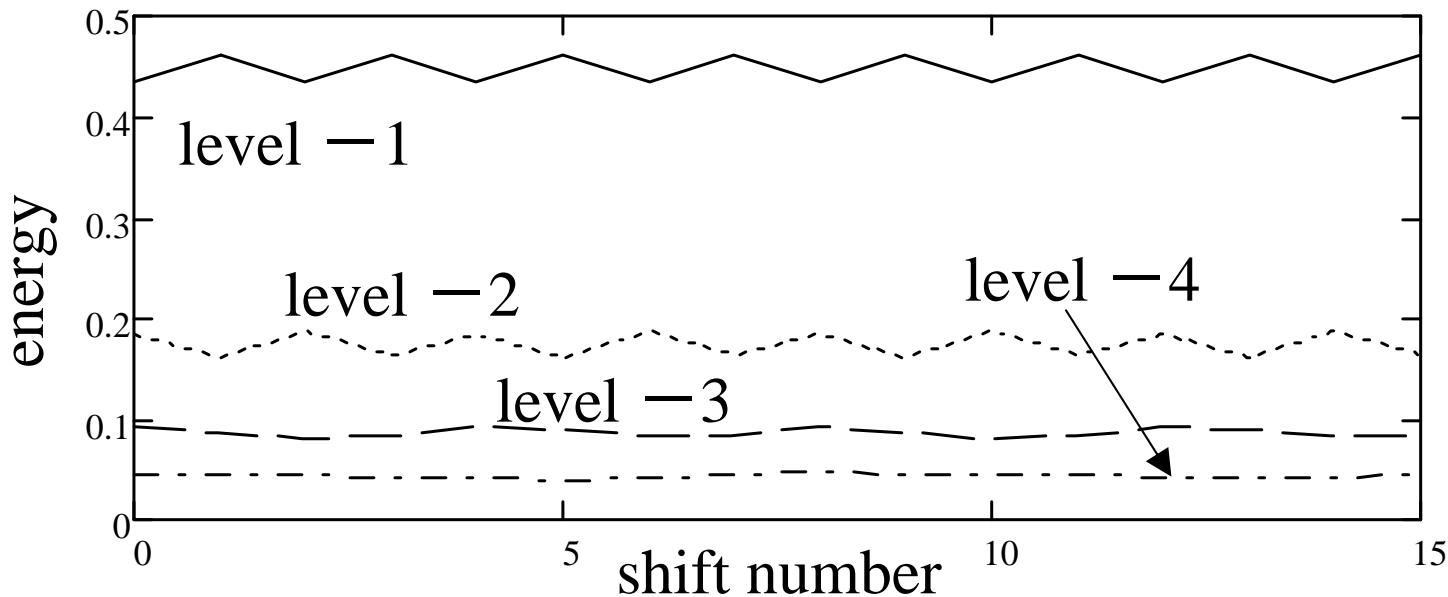
The fluctuation of impulse response energy of each level: (max 70%)

These are the lack of translation invariance

The impulse response of the CDWT by the RI-Daubecihes 6 wavelet



The impulse response of the level -3

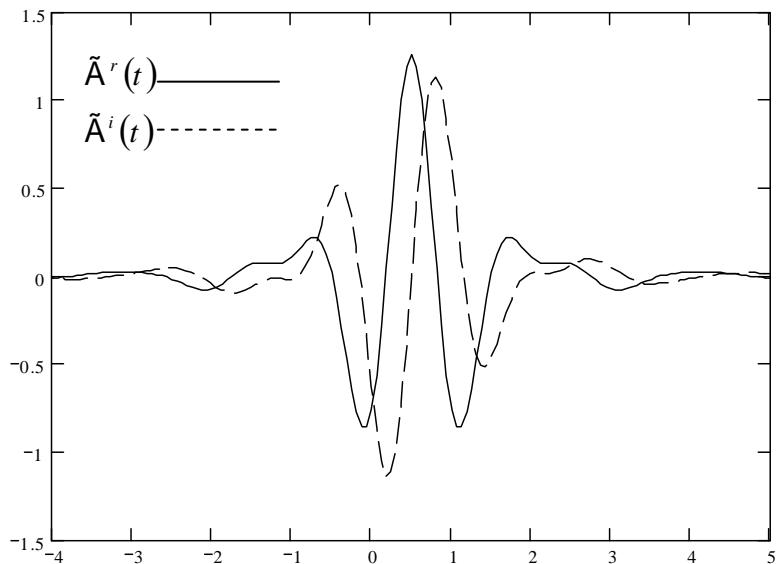


The fluctuation of impulse response energy of each level: (max 7.5%)

These are approximate translation invariance

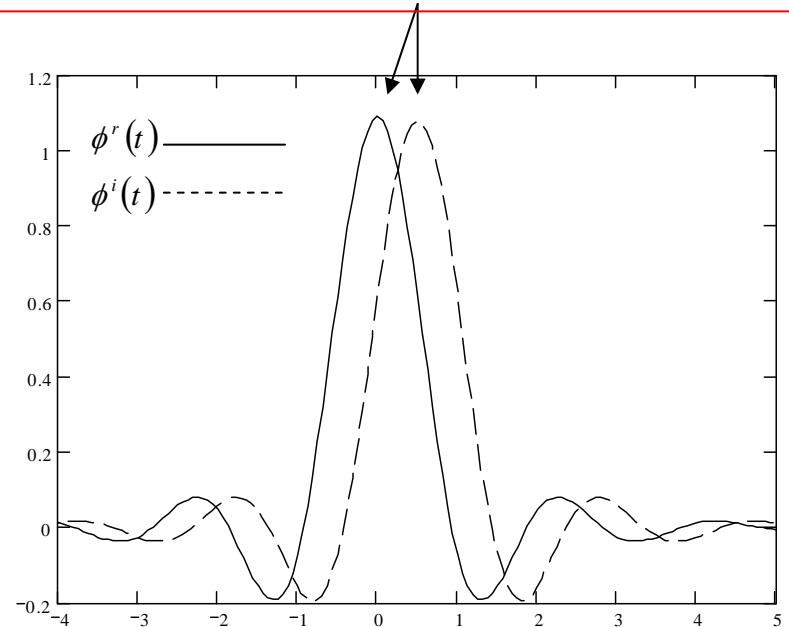
What is Reason of Energy Changing ?

A litter difference between scaling functions



Mother wavelet

RI-O-Spline wavelet

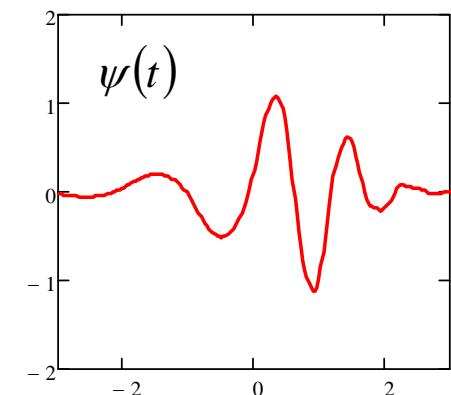


Scaling function

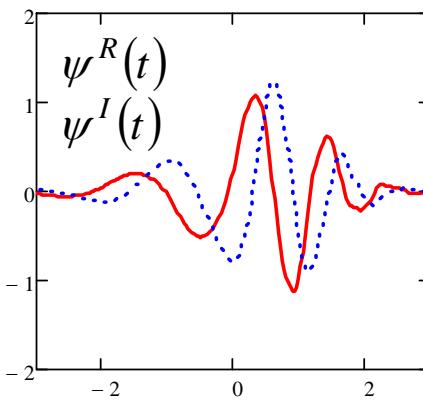
4.3.2 Requirement condition (2)

For scaling functions, their *shapes* of the real and imaginary components are completely same and *positions* are different from 1/2 sample

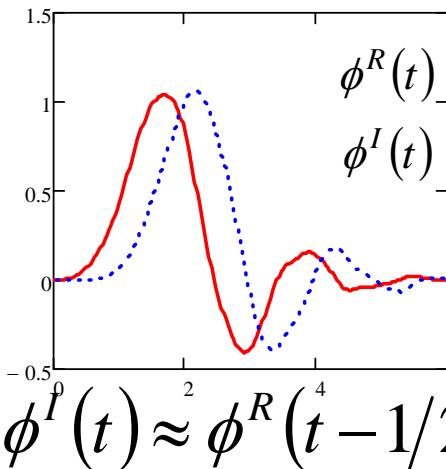
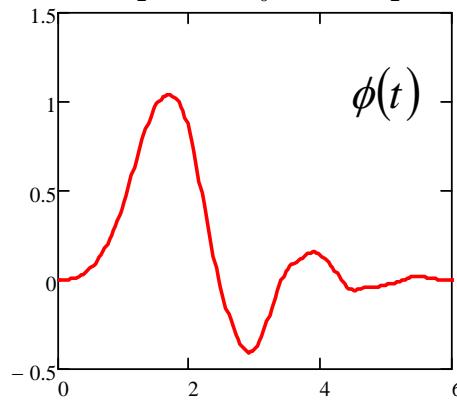
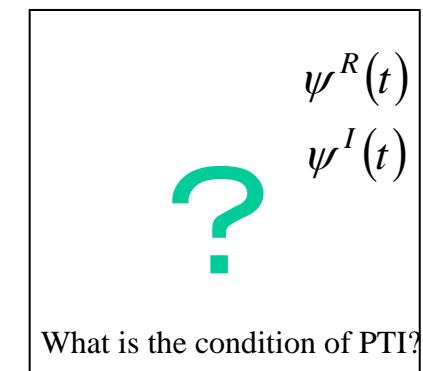
The real type wavelet



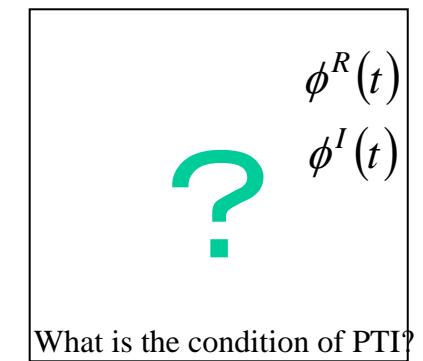
The Hilbert transform pair



The condition of PTI



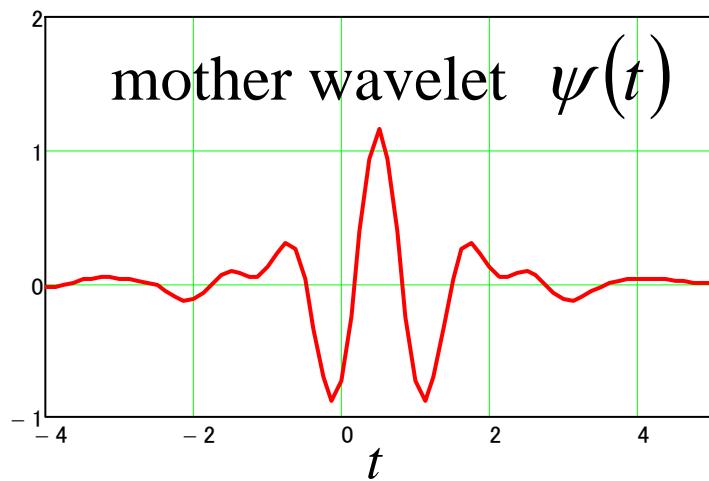
$$\phi^I(t) \approx \phi^R(t - 1/2)$$



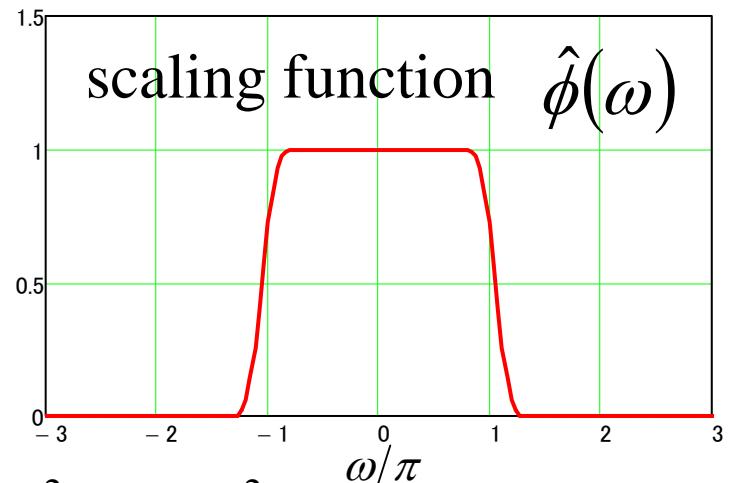
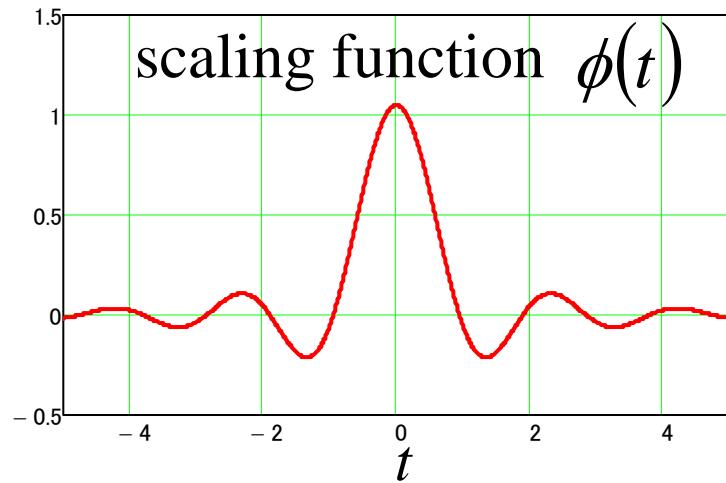
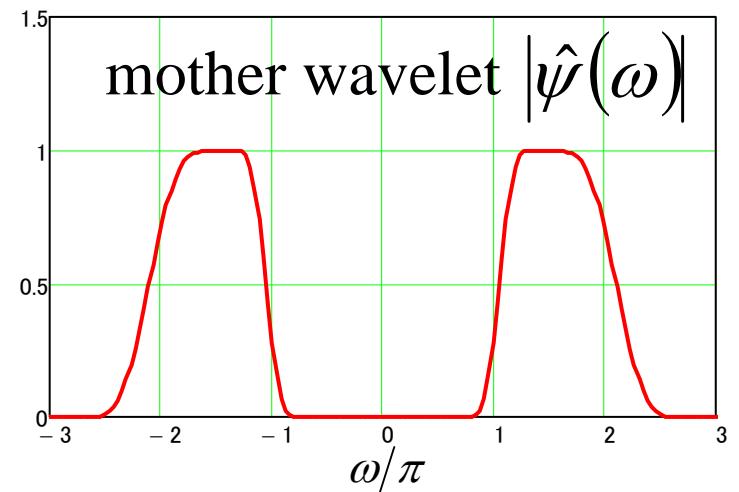
$$\phi^I(t) = \phi^R(t - 1/2)$$

1) Meyer wavelet (orthogonal & real type wavelet)

Time domain



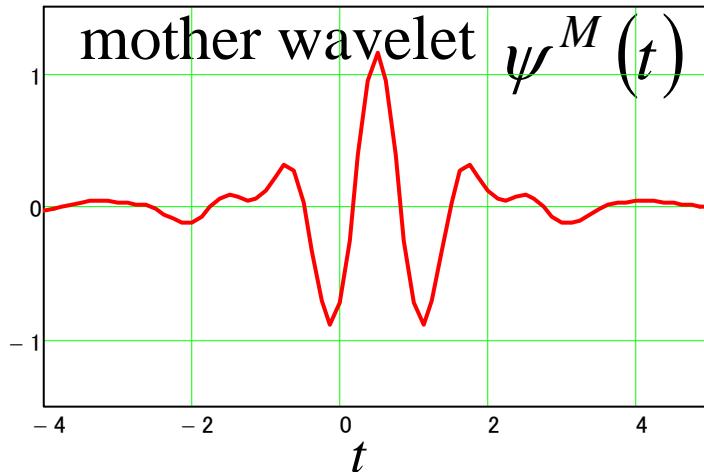
Frequency domain



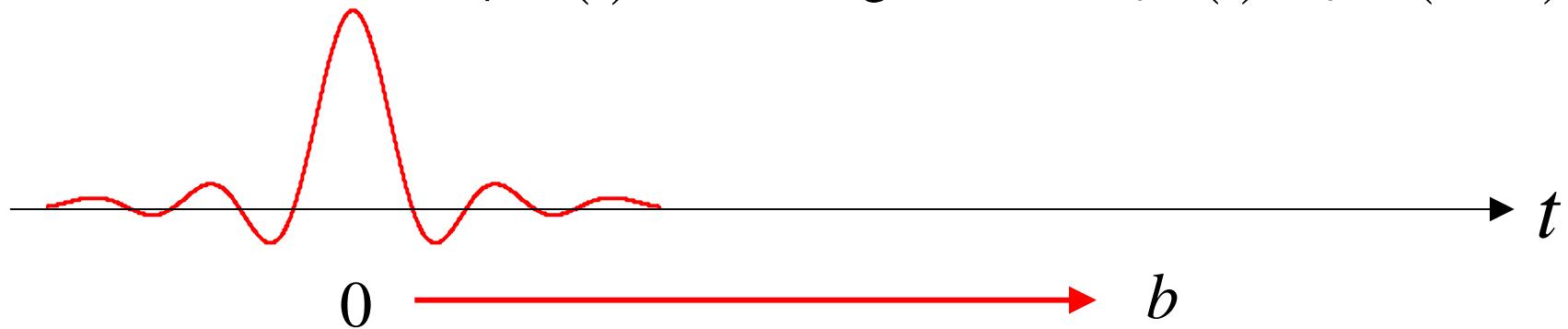
$$|\hat{\phi}(\omega - 2\pi)|^2 + |\hat{\phi}(\omega)|^2 = 1, \quad 2\pi/3 < \omega < 4\pi/3$$

The PTI complex wavelet is based on Meyers wavelet so let's introduces the Meyer wavelet.

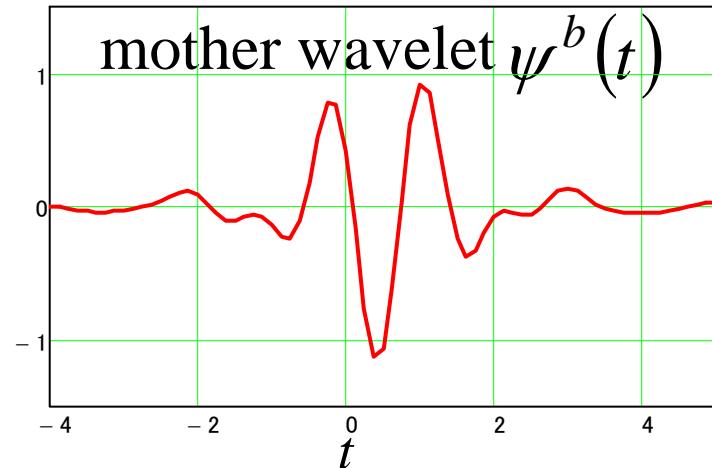
The excellent characteristics of the scaling function of the Meyer wavelet



scaling function $\phi^M(t)$



NEW!



scaling function $\phi^b(t) = \phi^M(t - b)$

The scaling function of the Mayer wavelet, which is arbitrarily translated, also becomes a new scaling function of the new wavelet.

2) PTI CDWTの設計

The condition of Perfect Translation Invariance (PTI) :

$\phi^R(t)$ and $\phi^I(t)$ have the same shapes with the 1/2-sample distance.

$$\phi^I(t) = \phi^R(t - 1/2)$$

This PTI condition is achieved by the characteristics of Meyer wavelet

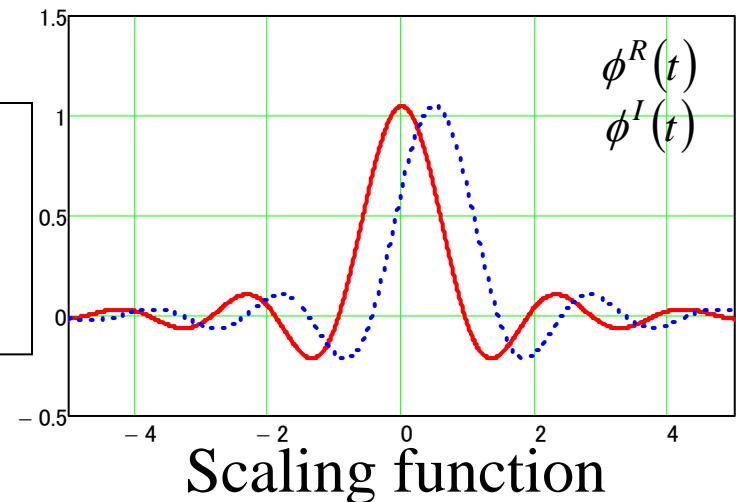
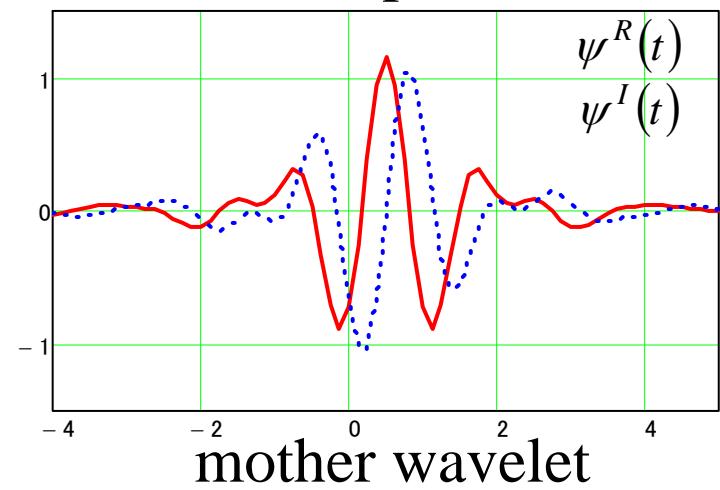
$$\phi^R(t) = \phi^b(t) = \phi^M(t - b)$$

$$\phi^I(t) = \phi^{b+1/2}(t) = \phi^M(t - (b + 1/2))$$

The translated scaling functions of the Meyer wavelet

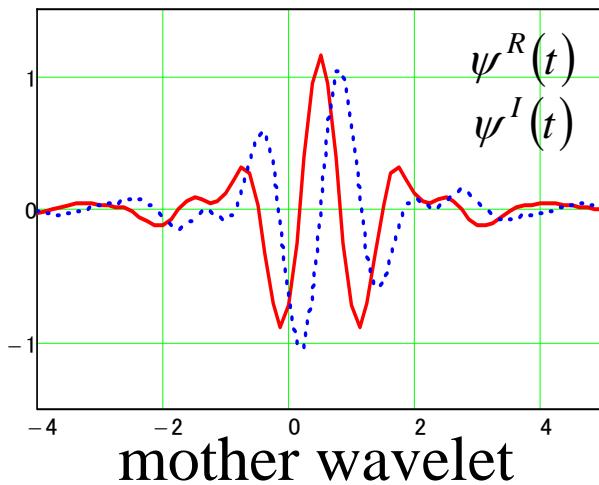
$$p_n^b = \phi^M\left(\frac{n-b}{2}\right)$$

The example of $b = 0$



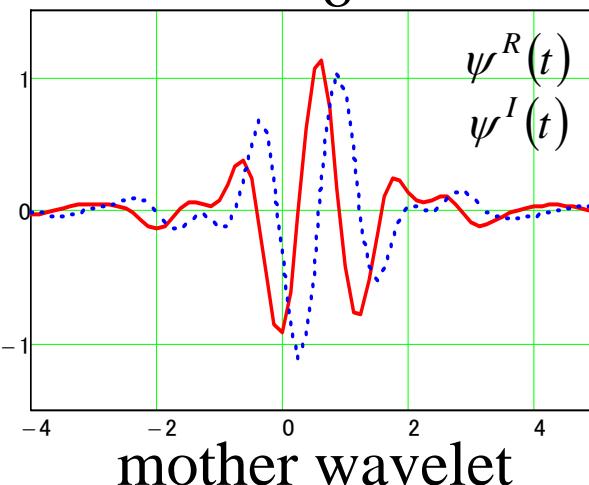
The variation of the PTI complex wavelet 1

$$b = 0$$



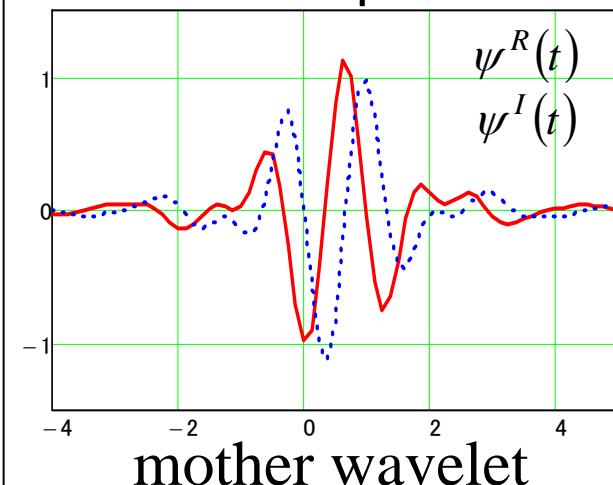
mother wavelet

$$b = \frac{1}{8}$$



mother wavelet

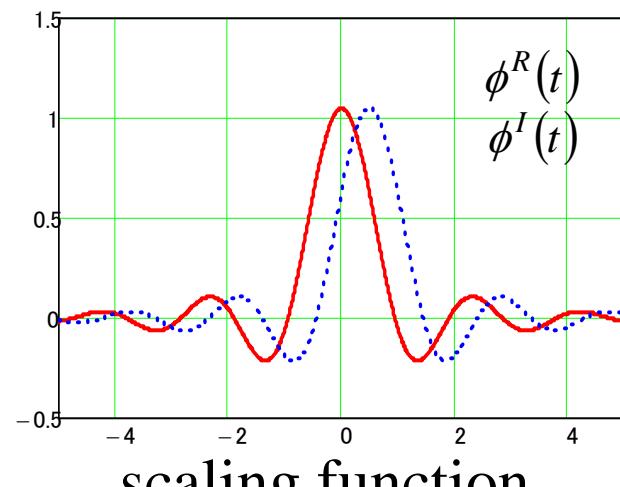
$$b = \frac{1}{4}$$



mother wavelet

$$\phi^R(t)$$

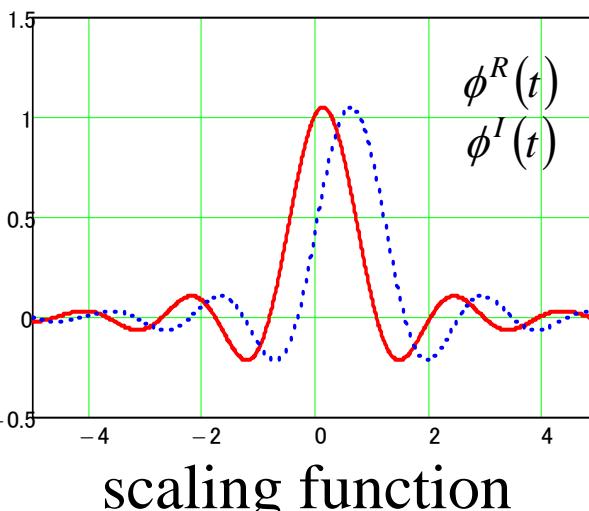
$$\phi^I(t)$$



scaling function

$$\phi^R(t)$$

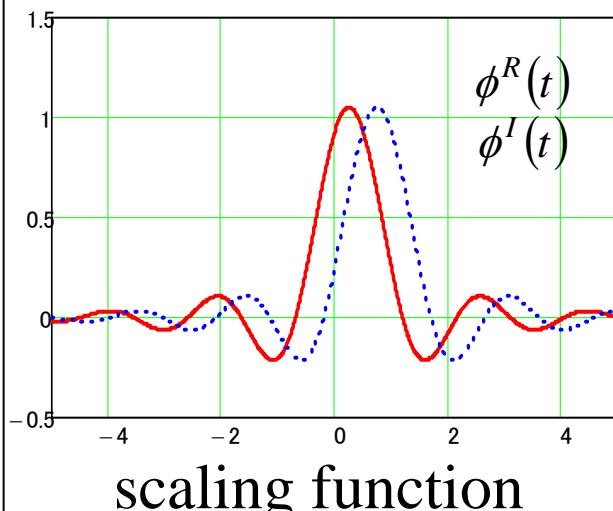
$$\phi^I(t)$$



scaling function

$$\phi^R(t)$$

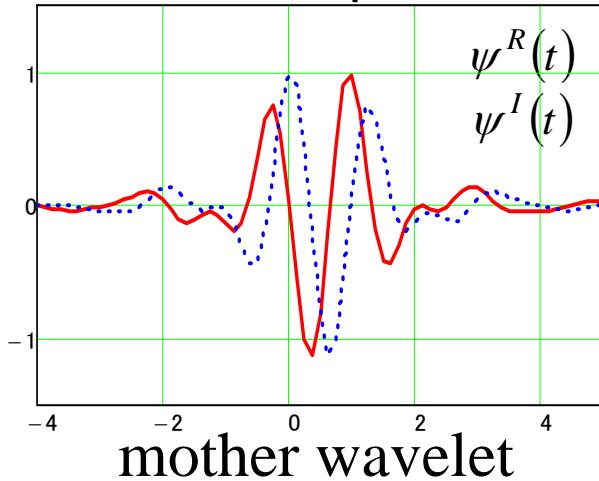
$$\phi^I(t)$$



scaling function

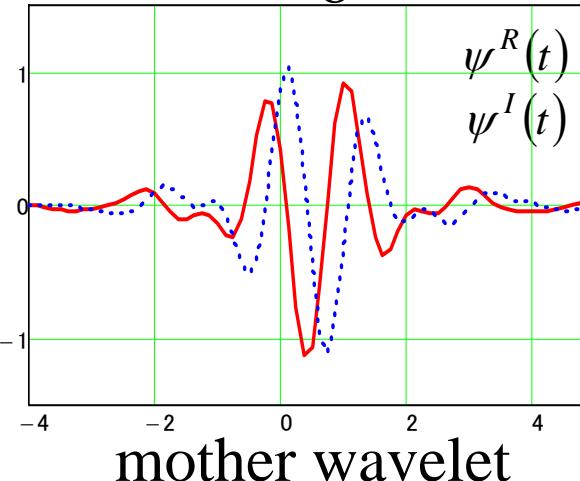
The variation of the PTI complex wavelet 3

$$b = \frac{3}{4}$$



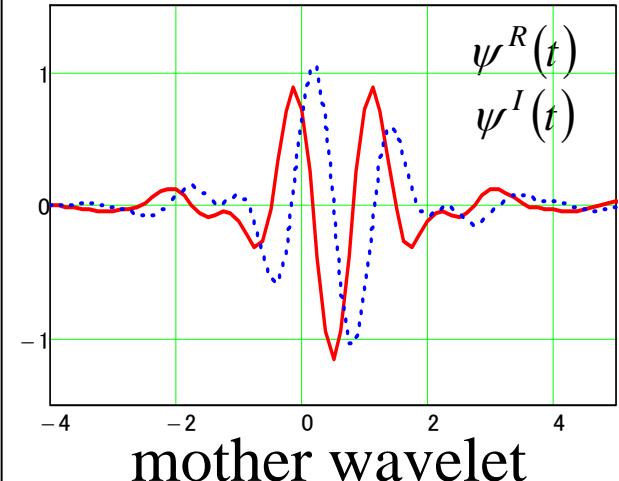
mother wavelet

$$b = \frac{7}{8}$$



mother wavelet

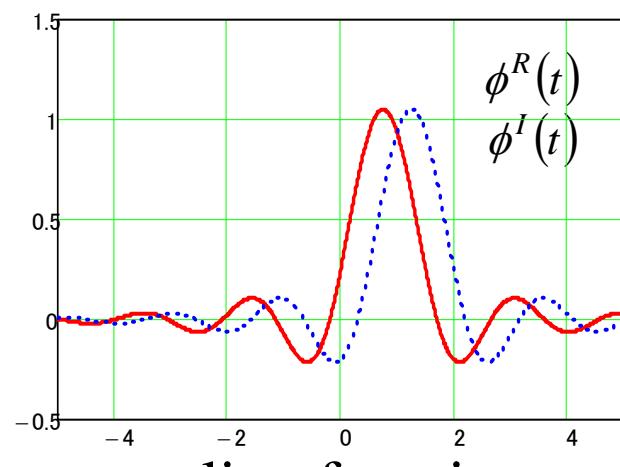
$$b = 1$$



mother wavelet

$$\phi^R(t)$$

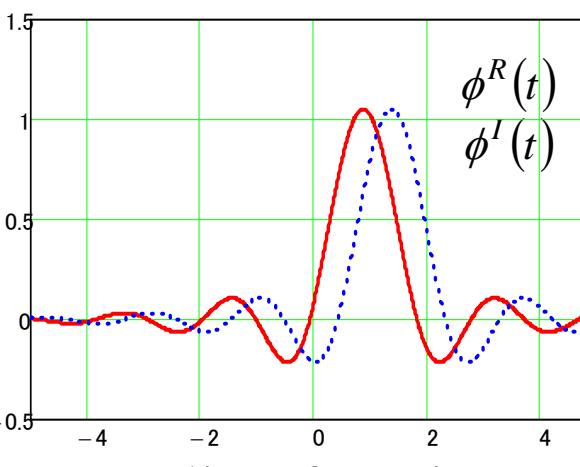
$$\phi^I(t)$$



scaling function

$$\phi^R(t)$$

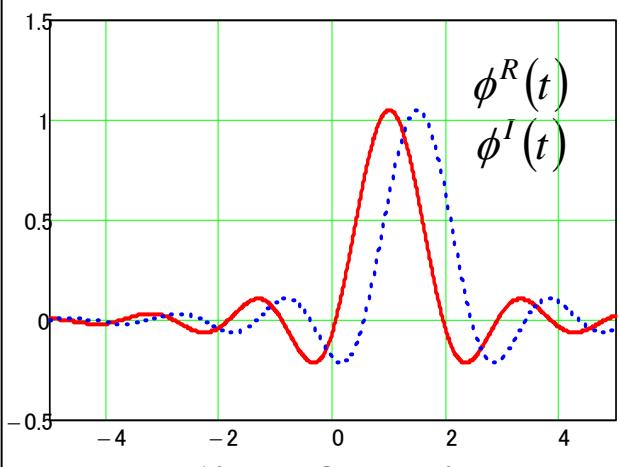
$$\phi^I(t)$$



scaling function

$$\phi^R(t)$$

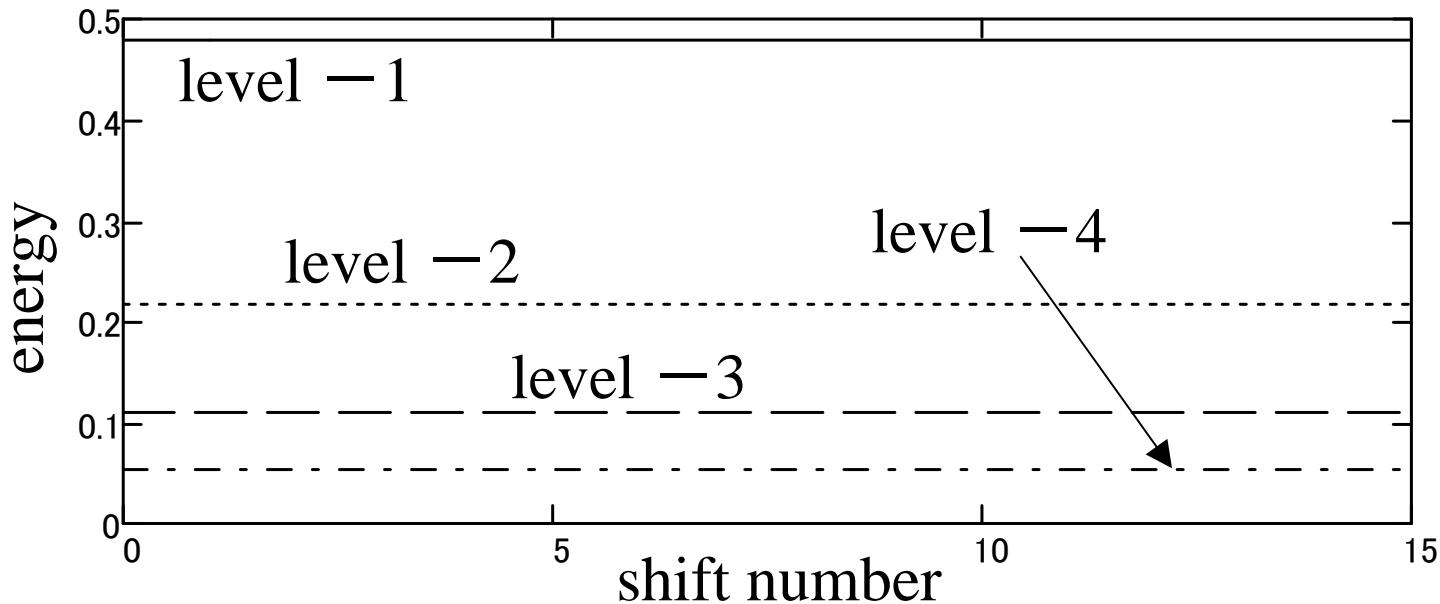
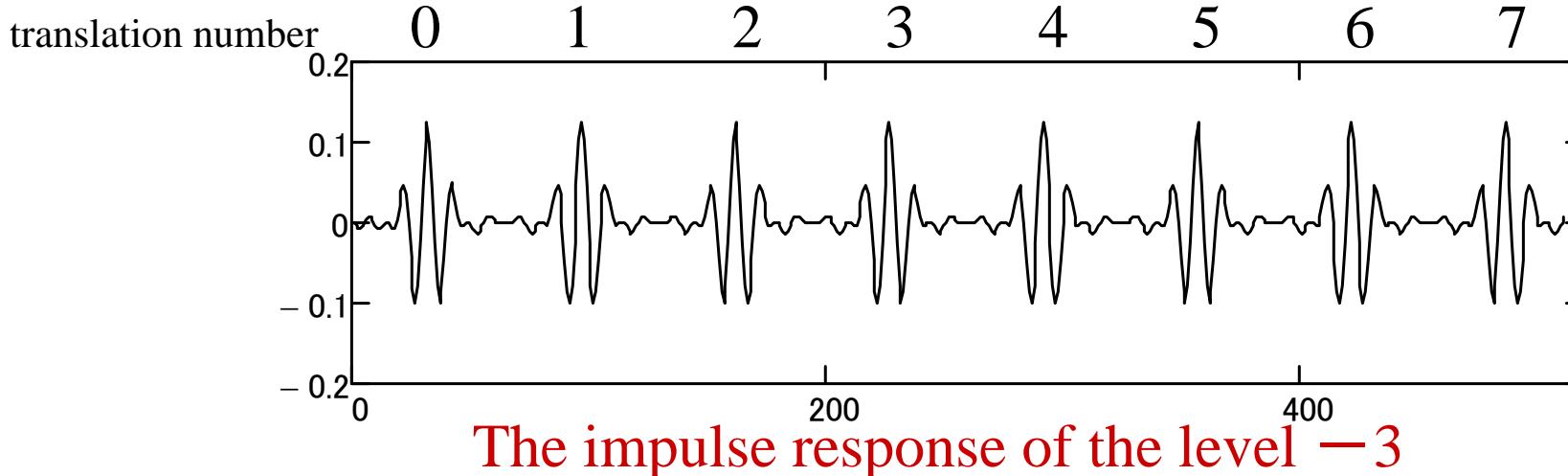
$$\phi^I(t)$$



scaling function

The shape of the PTI complex wavelet is 2-periodic of b .

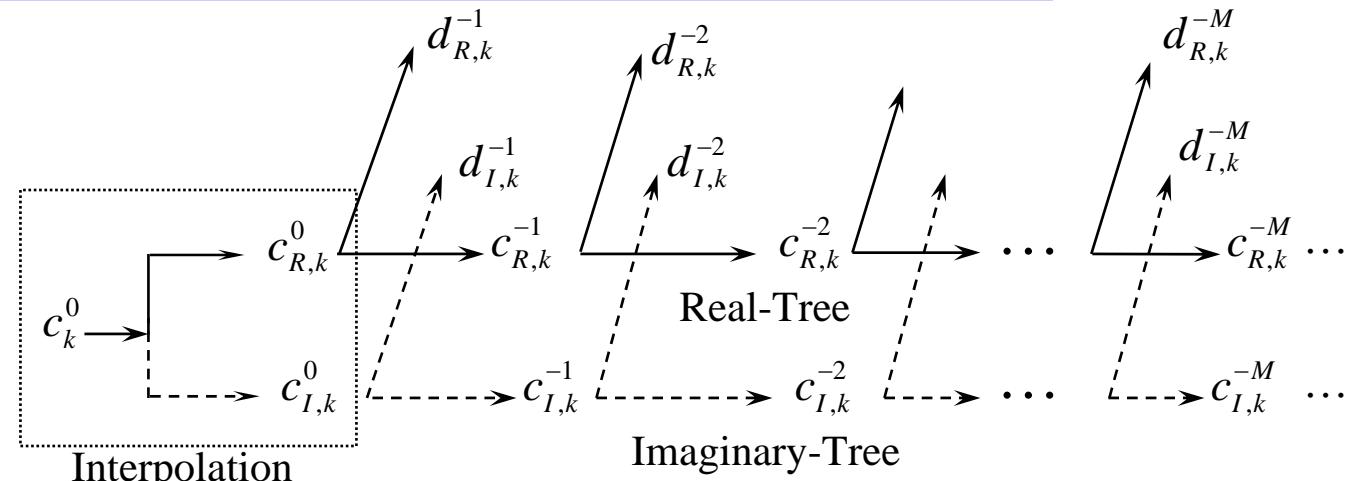
The impulse response of the CDWT by the PTI complex wavelet



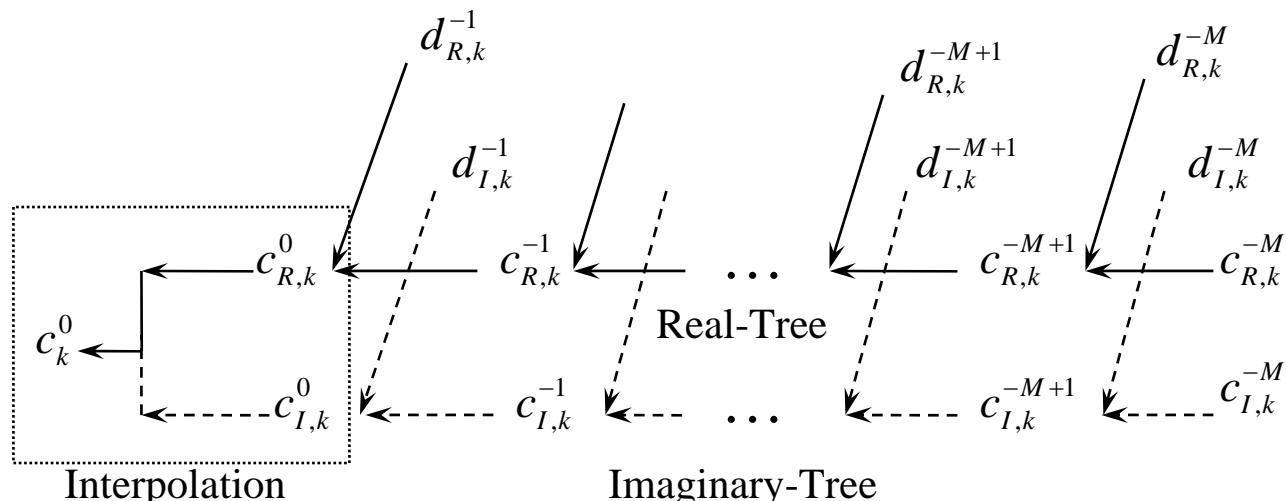
The fluctuation of impulse response energy of each level: (max 0.0001%)

4.4 New Calculation methods for CDWT

1) Coherent Dual--Tree Algorithm (DTA)

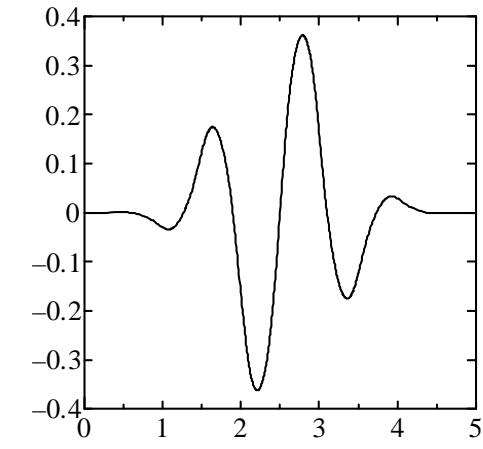


(a) Decomposition Tree

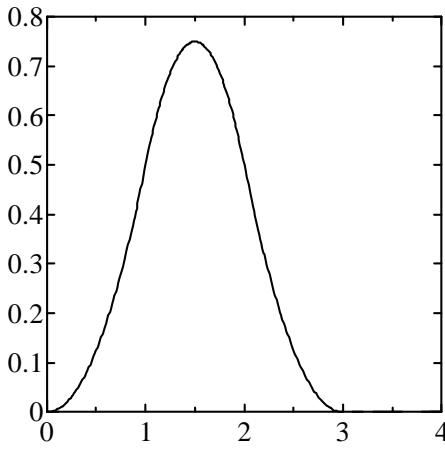


(b) Reconstruction Tree

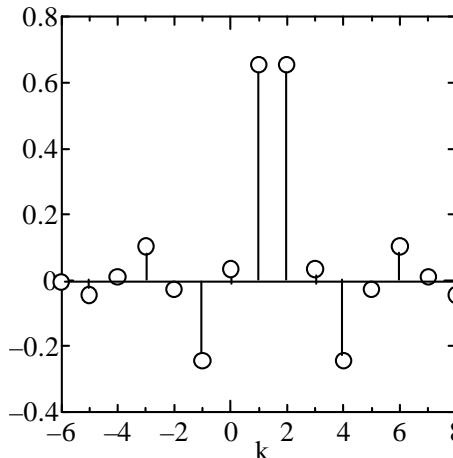
What is half sample delay?



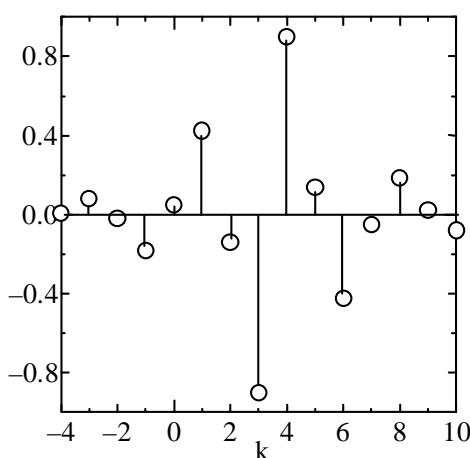
(a)Wavelet



(b)Scaling function

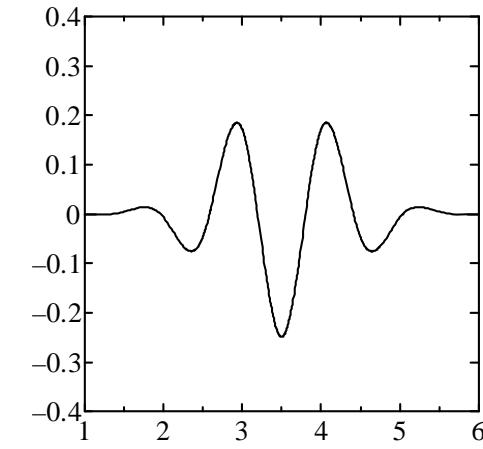


(c) a_k

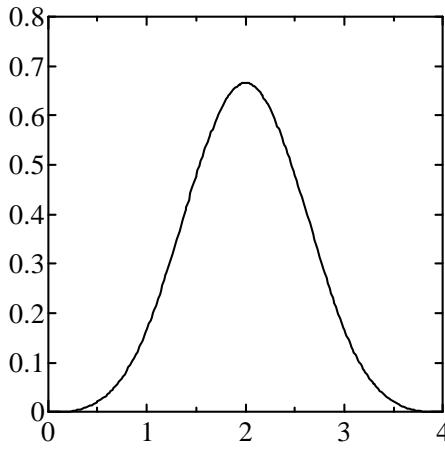


(d) b_k

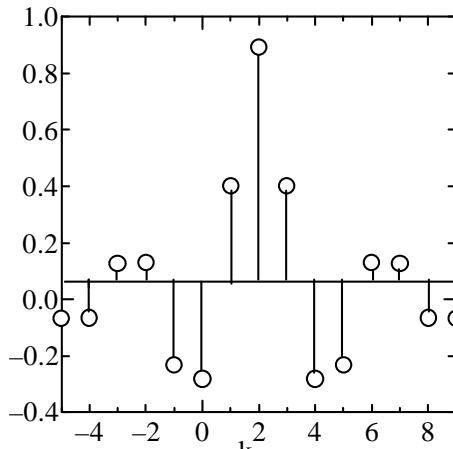
(a) $m=3$ Spline Wavelet



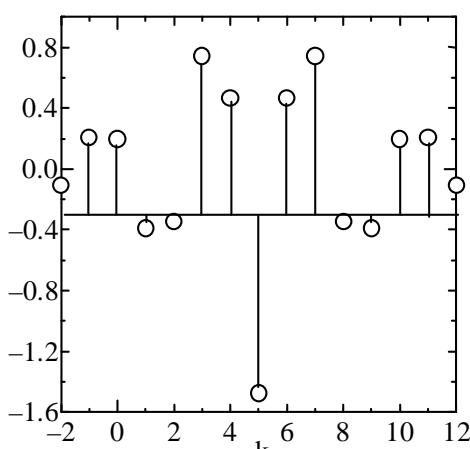
(a)Wavelet



(b)Scaling function



a_k



(d) b_k

(b) $m=4$ Spline waveley

There is a half sample delay problem between two wavelet

Q-shift Dual Tree Complex Wavelet Transform in 1-D Level 4

(By N. Kingsbury)

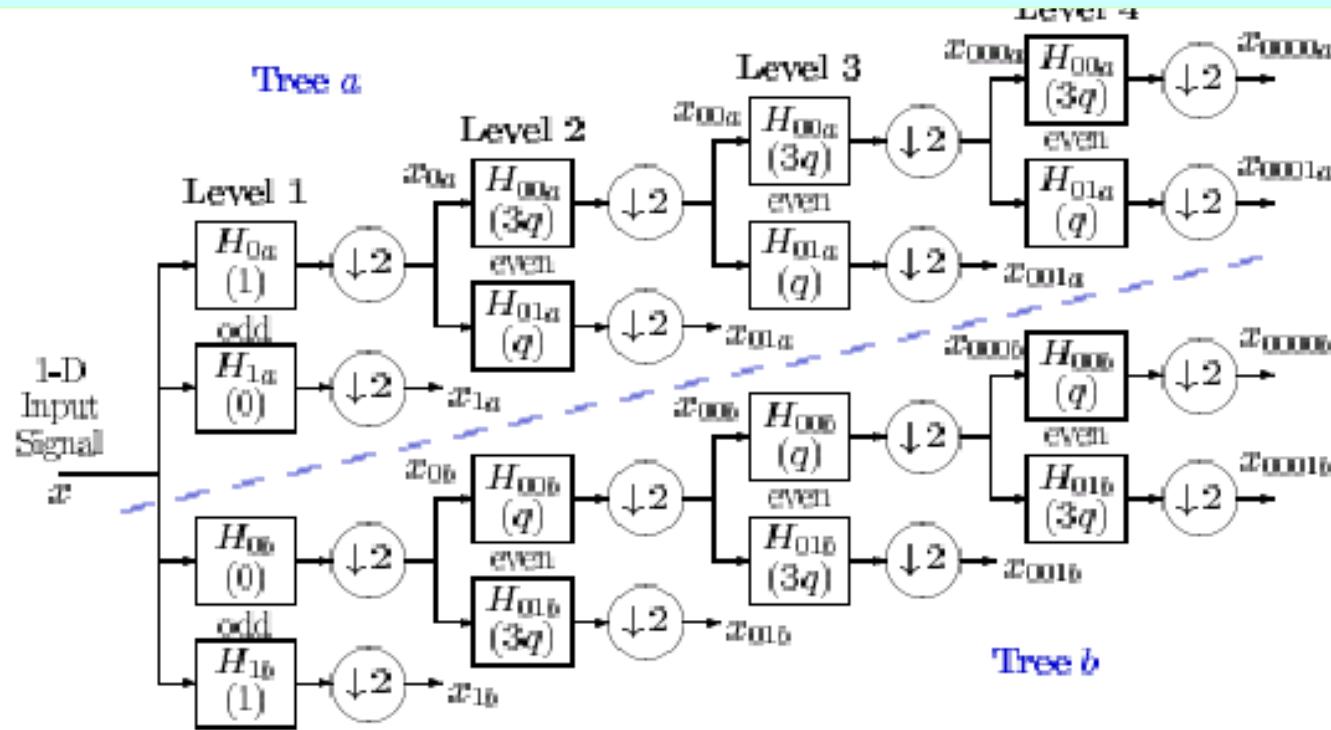


Figure: Dual tree of real filters for the Q-shift DWT, giving real and imaginary parts of complex coefficients from tree *a* and tree *b* respectively. Figures in brackets indicate the approximate delay for each filter, where $q=1/4$ sample period.

Drawback: designing filter with $q=1/4$ sample period is not easy

Interpolation

$$f(t) = \sum_l c_l^0 N_s(t-l)$$

$$c_k^0 = \sum_l f(l) \beta_{k-l}^S$$

$$c_{R,k}^0 = \sum_l c_l^0 K_{k-l}^R$$

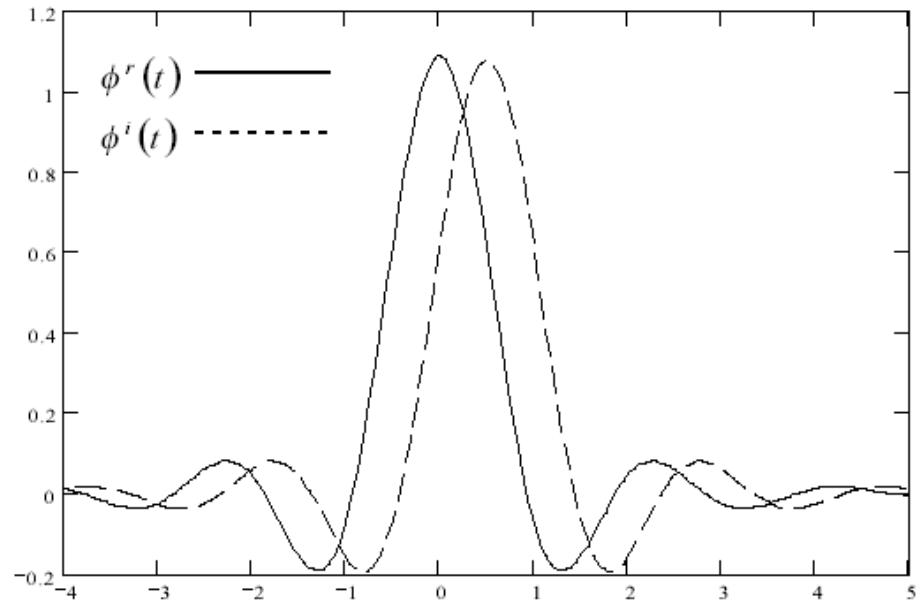
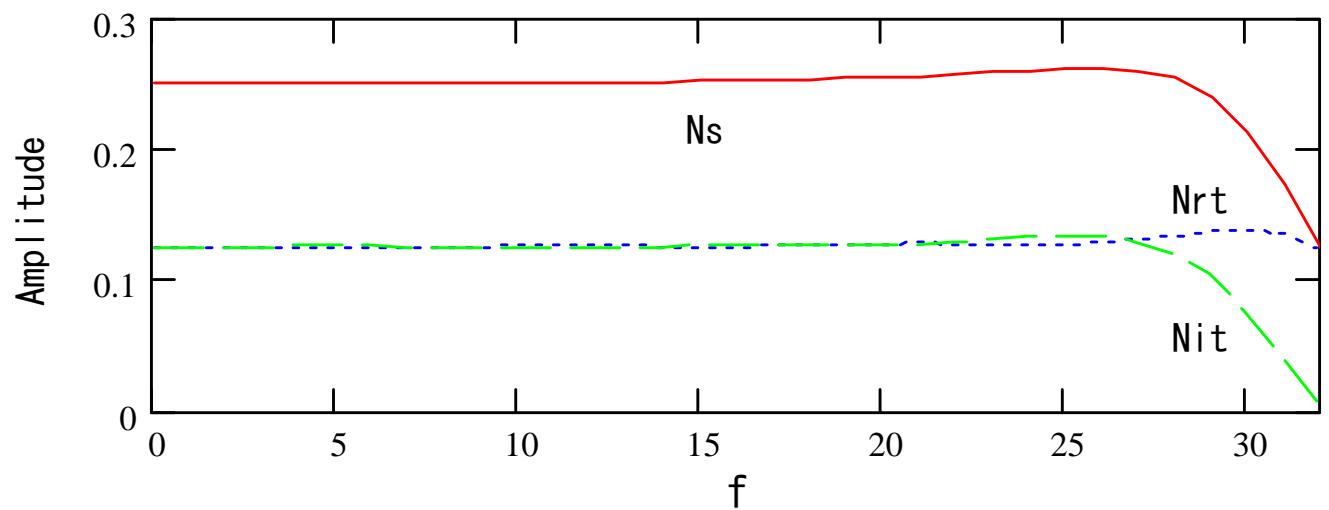
$$c_{I,k}^0 = \sum_l c_l^0 K_{k-l}^I$$

Synthesis scaling function:

$$N_s(t) = \sum_k K_k^R N_R(t-k) + \sum_k K_k^I N_I(t-k)$$

$$N_R(t) = N_{me}(t + me/2)$$

$$N_I(t) = N_{mo}(t + (mo-1)/2)$$



2) Results

(1) MRA (Mallat)

$$d_k^{j-1} = \sum_k a_{l-2k} c_k^j$$

$$c_k^{j-1} = \sum_k b_{l-2k} c_k^j$$

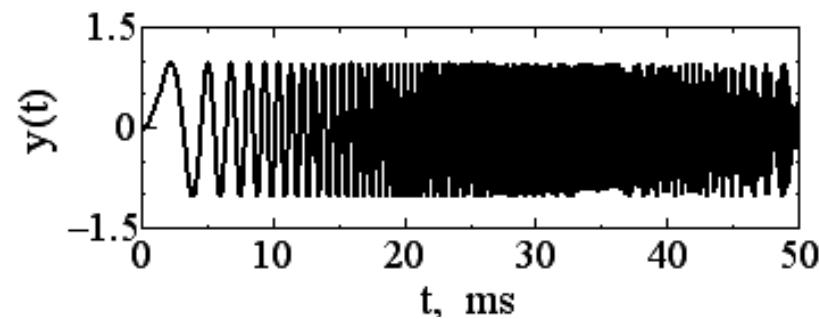
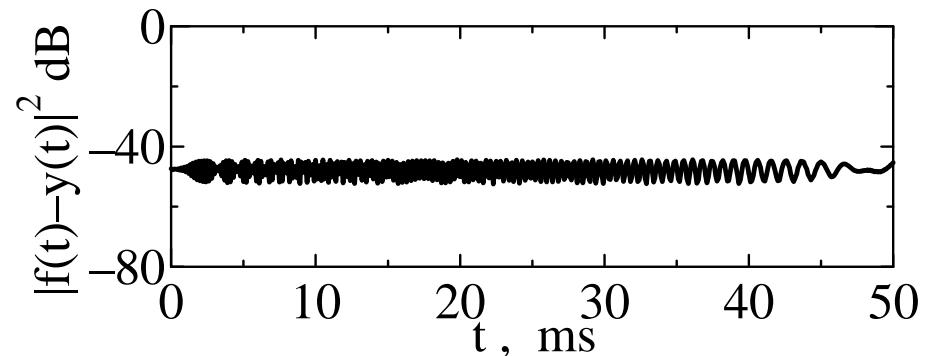
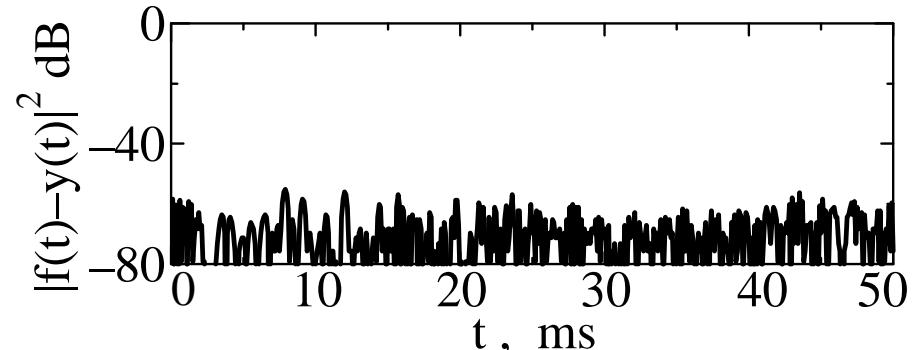


Fig.19 Chap signal



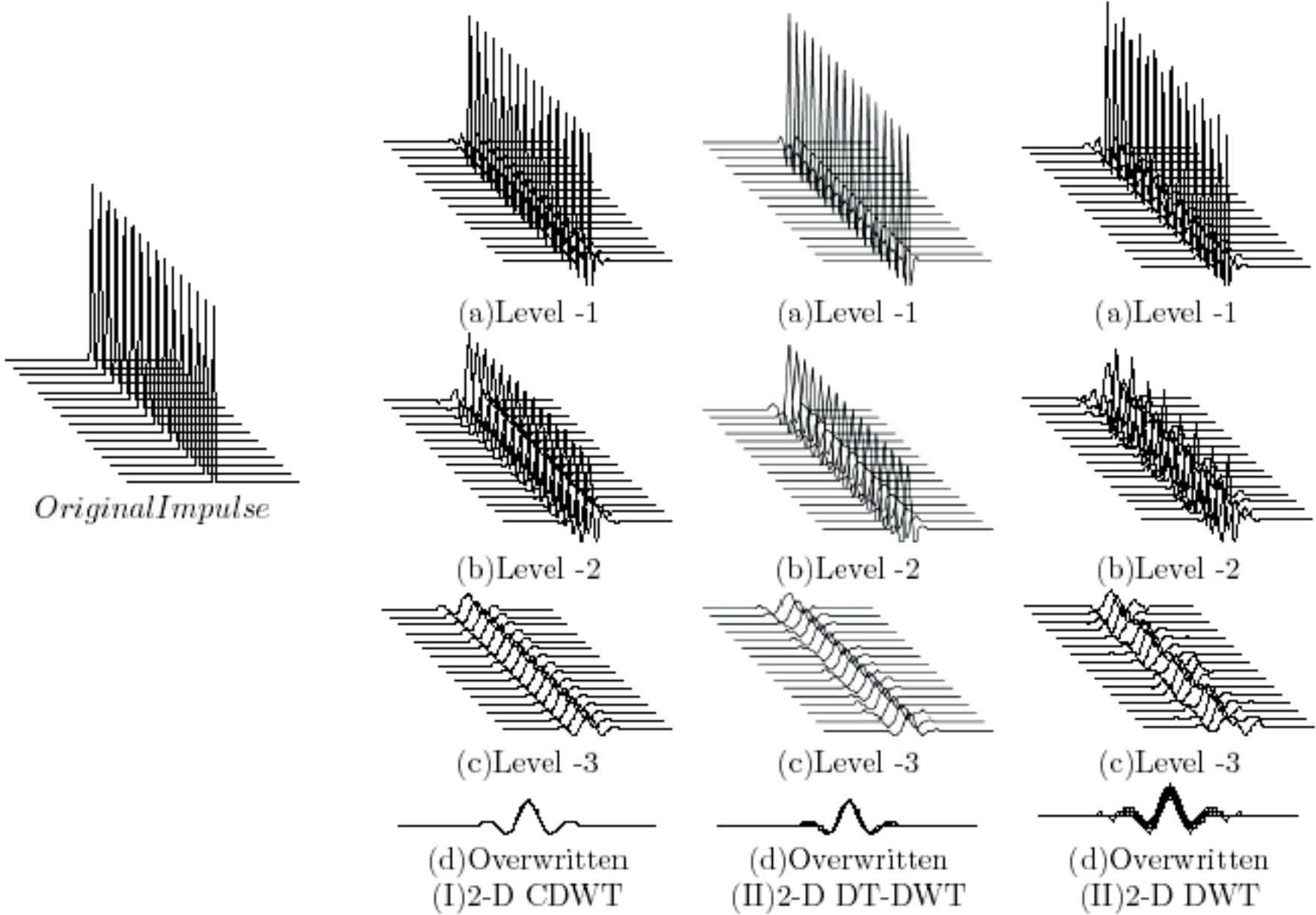
(a)Reconstructed error with FAFD



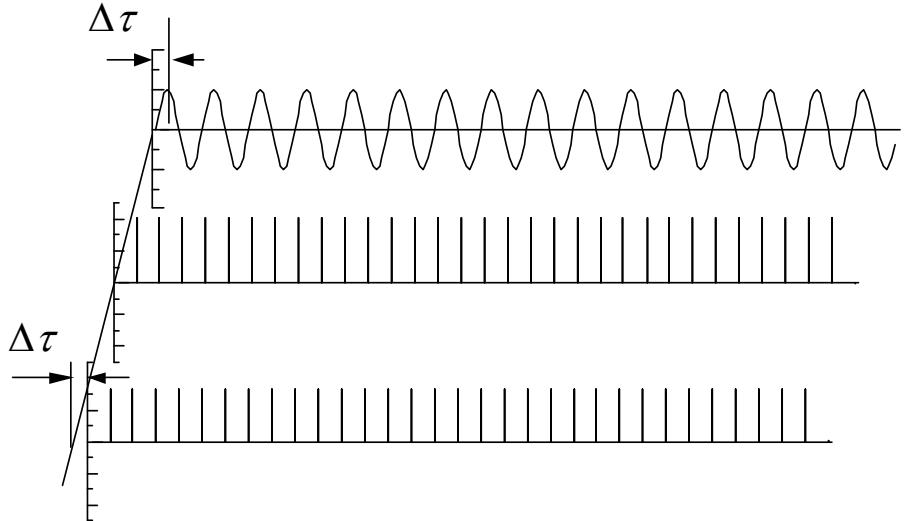
(b) Reconstructed error with CMRA

Fig.20 Reconstructed error using $m=3,4$ RI-Spline wavelet

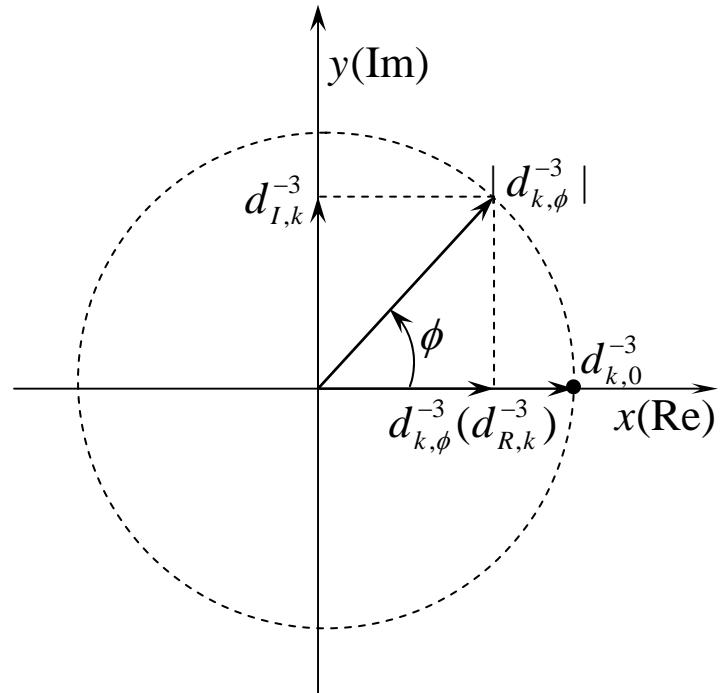
(2)Impulse response



(3) Sin signal

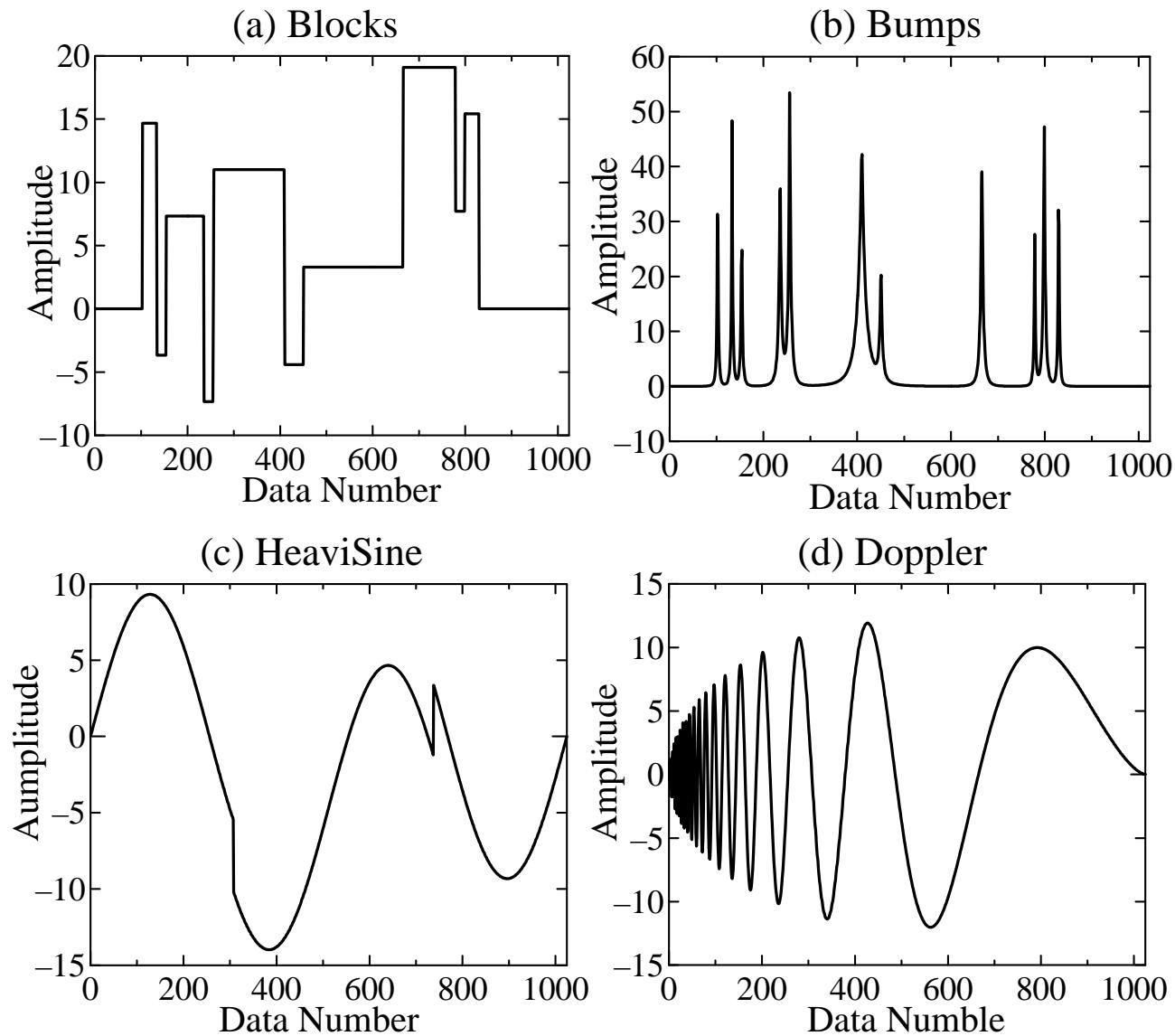


An example of wavelet coefficients in level -3 obtained by the RI- Spline wavelet, where signal is $f(k) = \sin(\pi k / 8)$

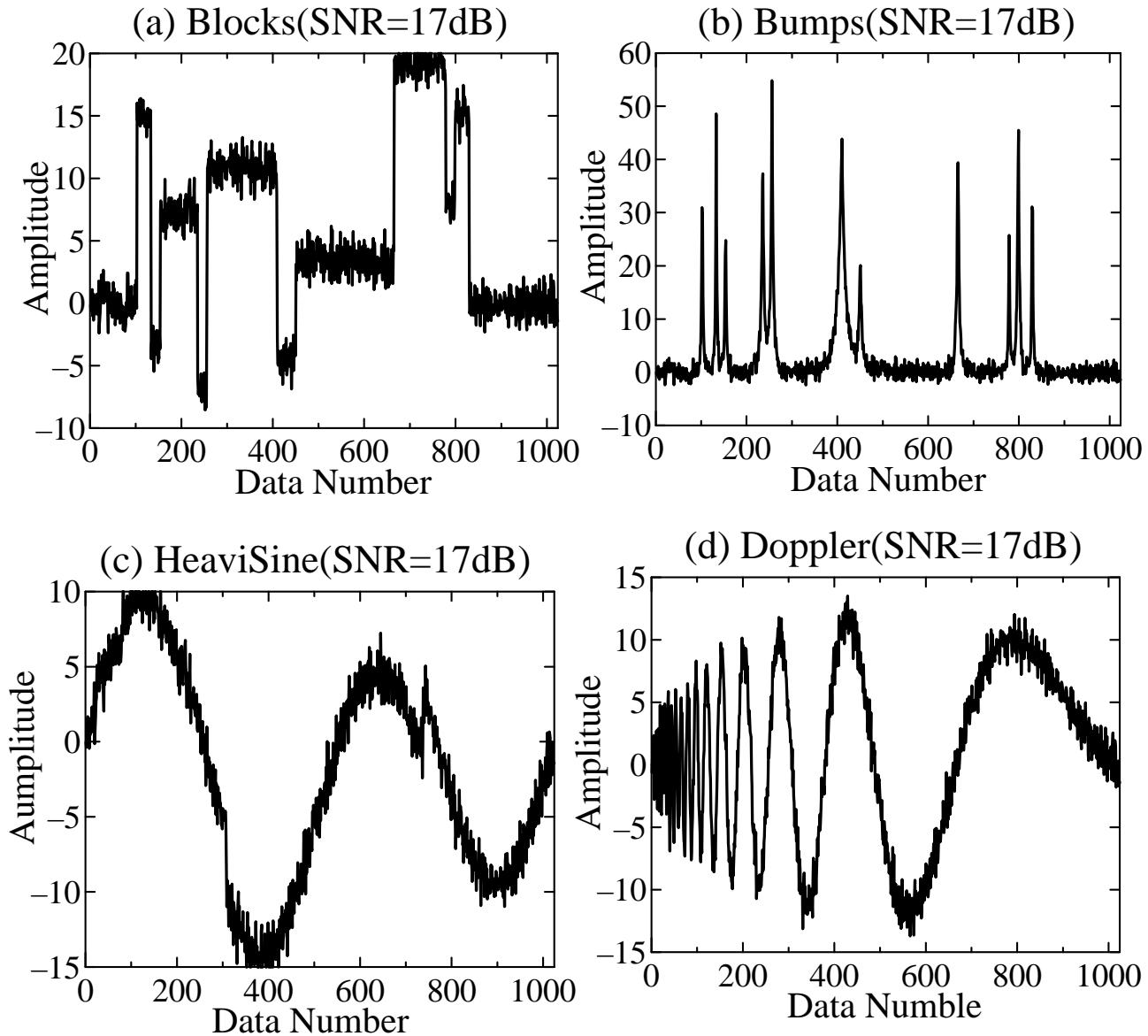


Influence of signal's phase

4.5 Examples of De-noising



4.5.1 Model signal de-noise



Influence of shifting

(MW is Daubichies 8)

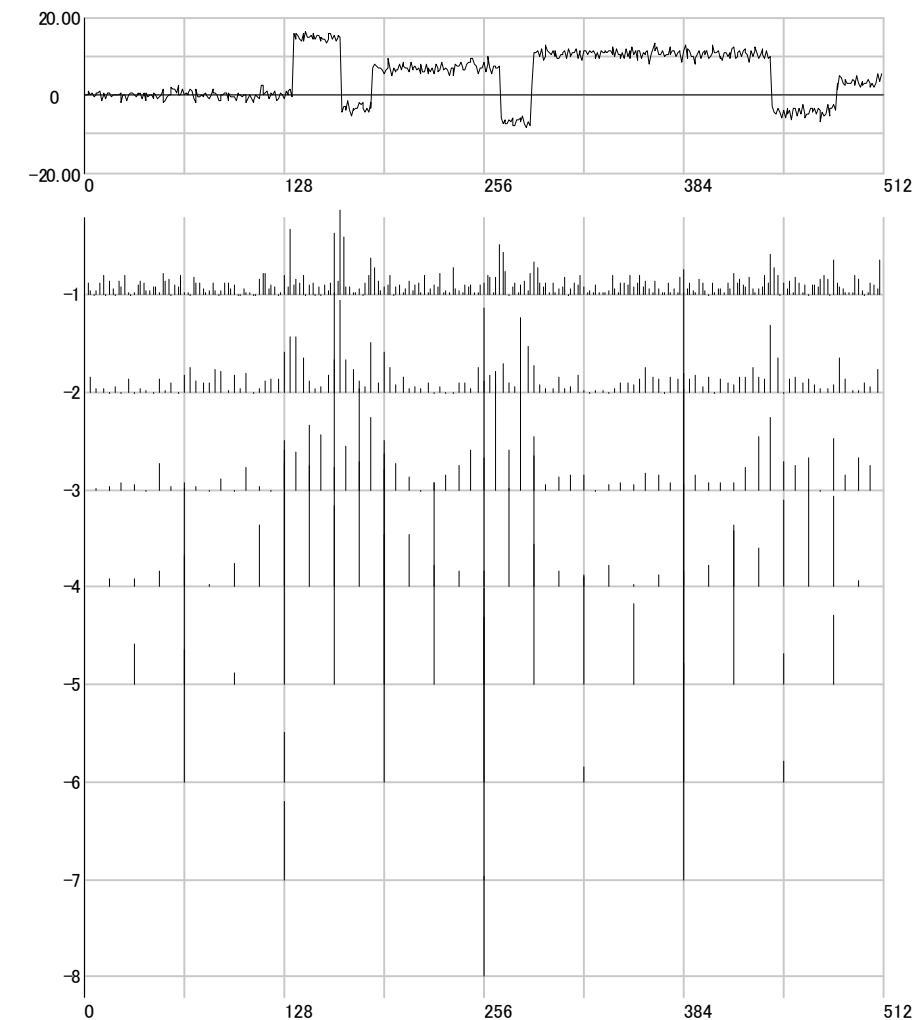


Fig.5 Original signal

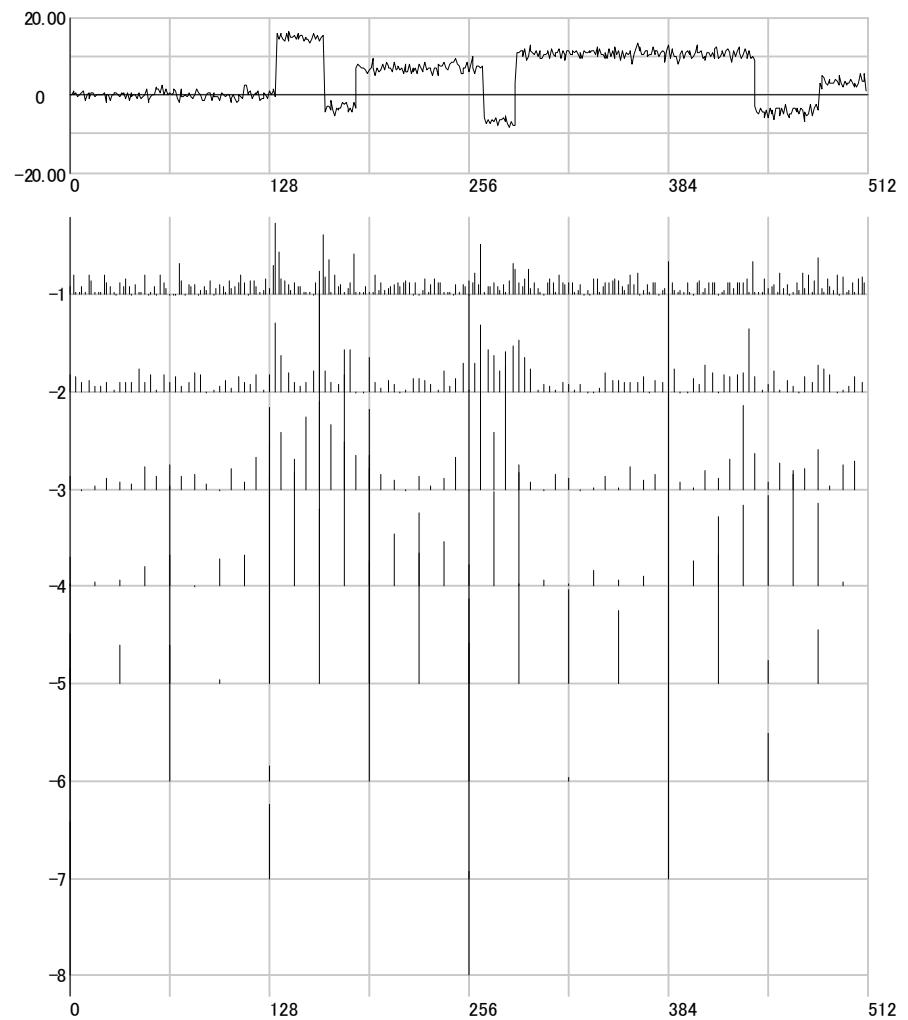


Fig.6 Signal with shift of one sample

Example of De-noising

MW is the RI-Spline Wavelet

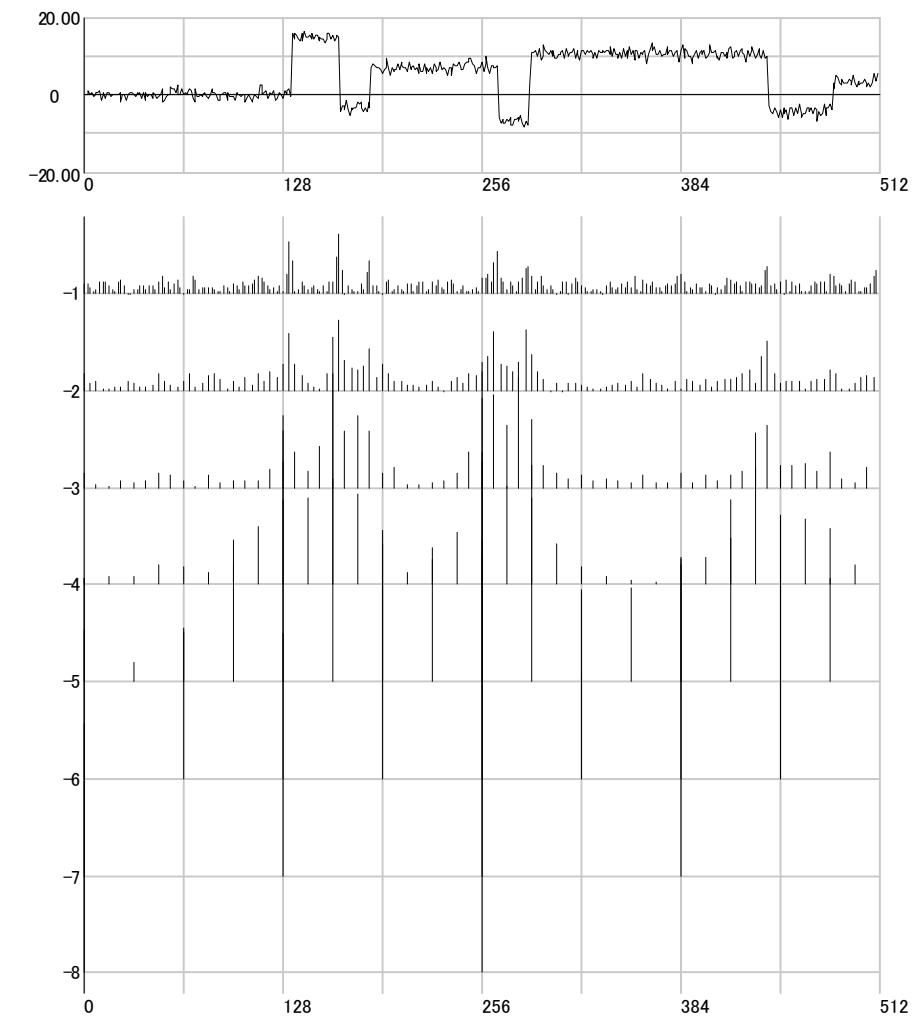


Fig.7 Original signal

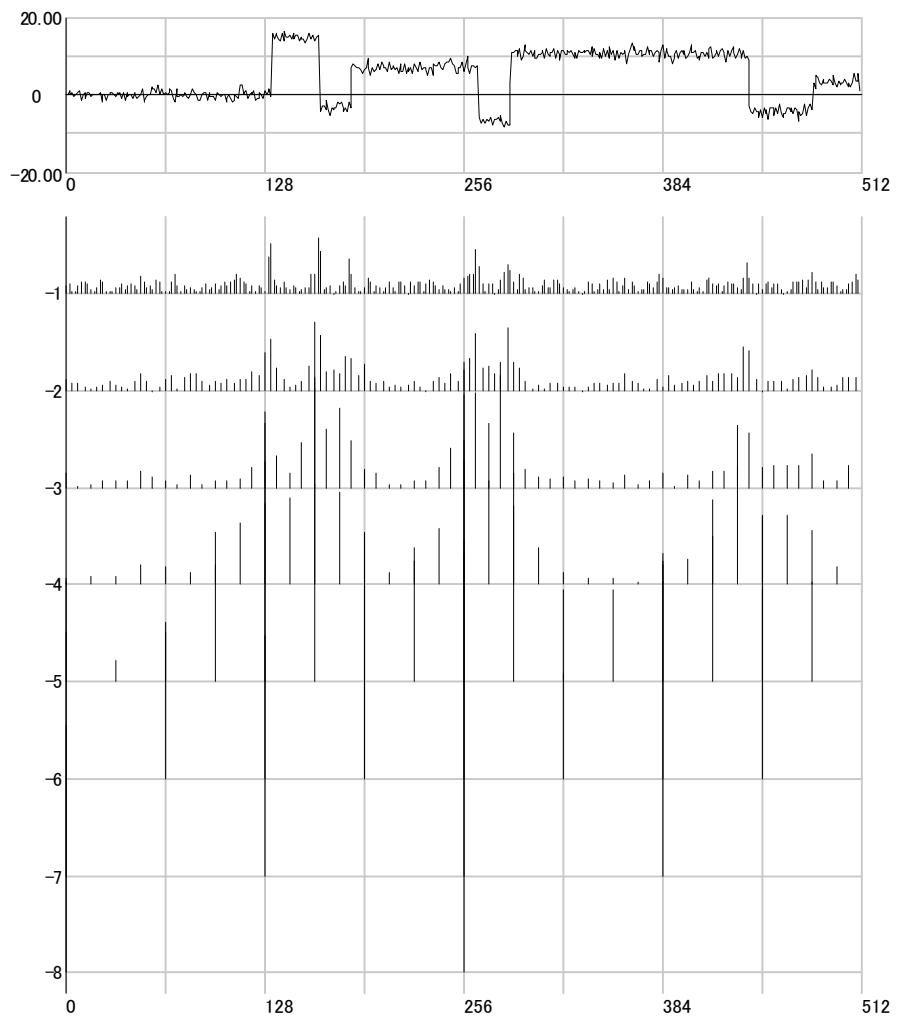


Fig.8 Signal with shift of one sample

Results of De-noising

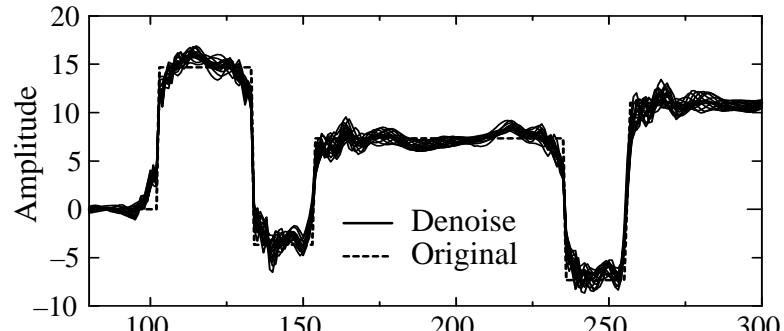
De-niosing:

1) Soft Thresholding

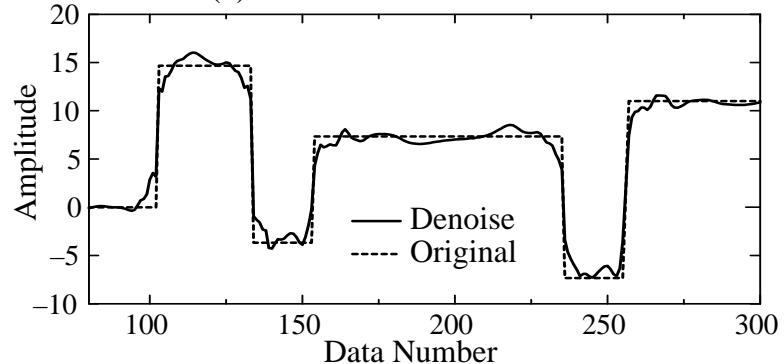
$$\hat{d} = \begin{cases} d - \lambda & (d > \lambda) \\ 0 & (d \geq \lambda) \end{cases}$$

$$(d = \sqrt{(d_{j,k}^r)^2 + (d_{j,k}^i)^2})$$

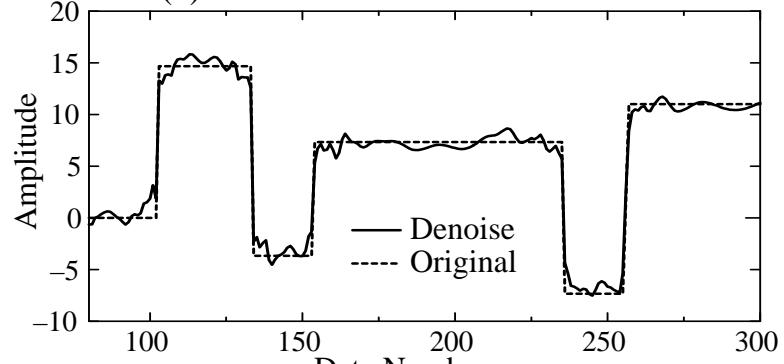
2) Result obtained by D8-TI is average value of 16 cycle data (0-15 sample Shift)



(a) Denoise result with D8

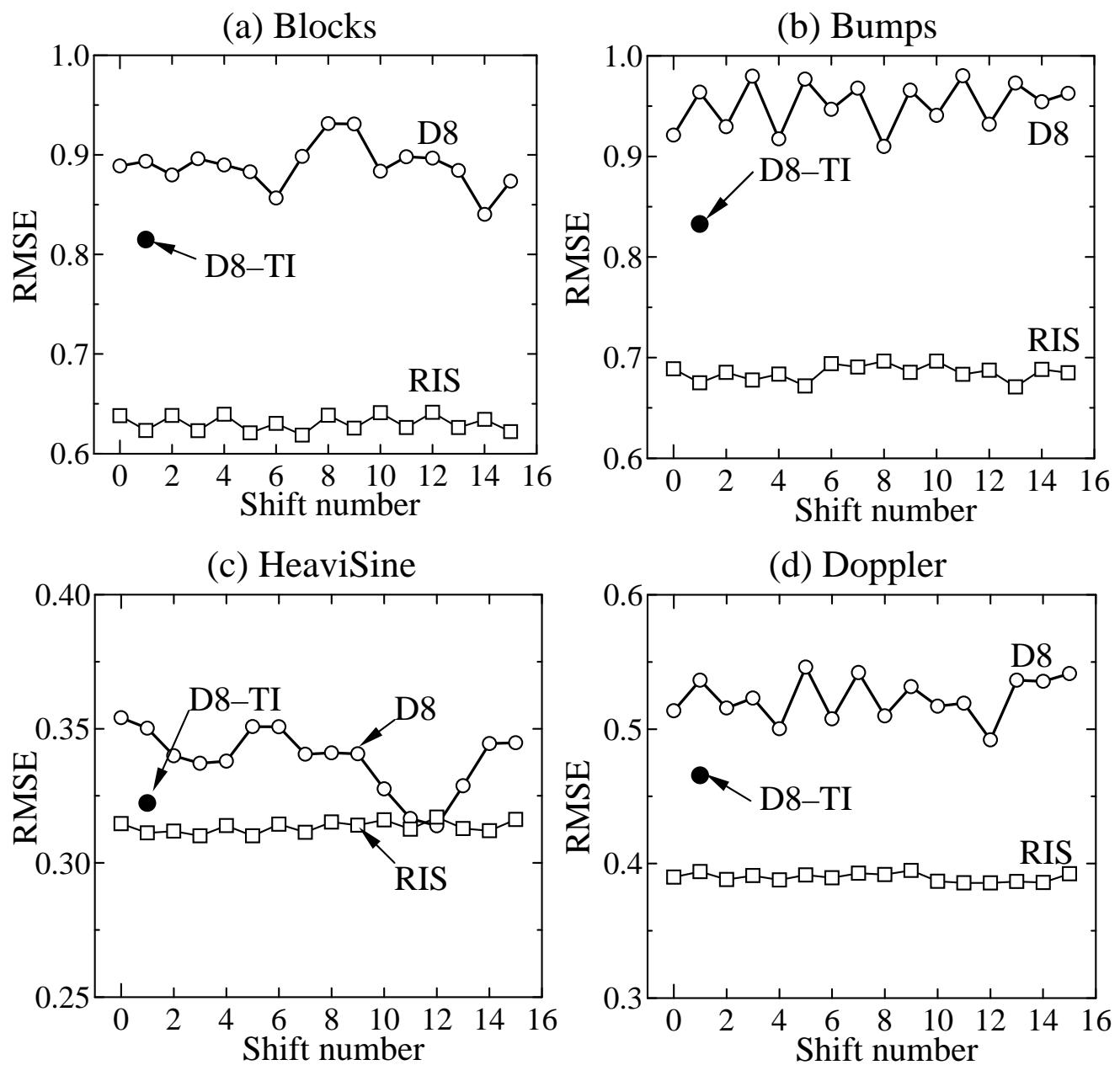


(b) Denoise result with D8-TI



(c) Denoise result with RIS

RMSE



4.5.2 De-noise of ECG

(Wavelet Shrikage)

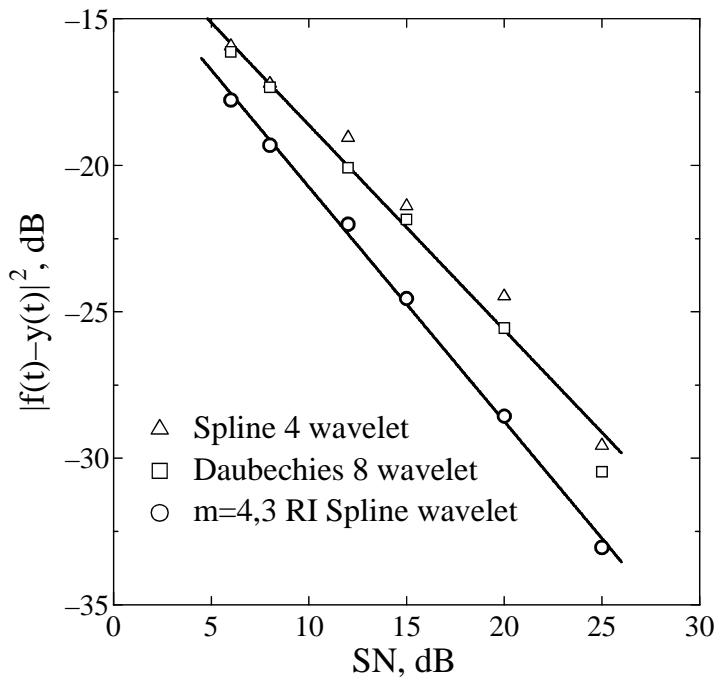


Fig.27 Transformation of the ECG wave after de-noise

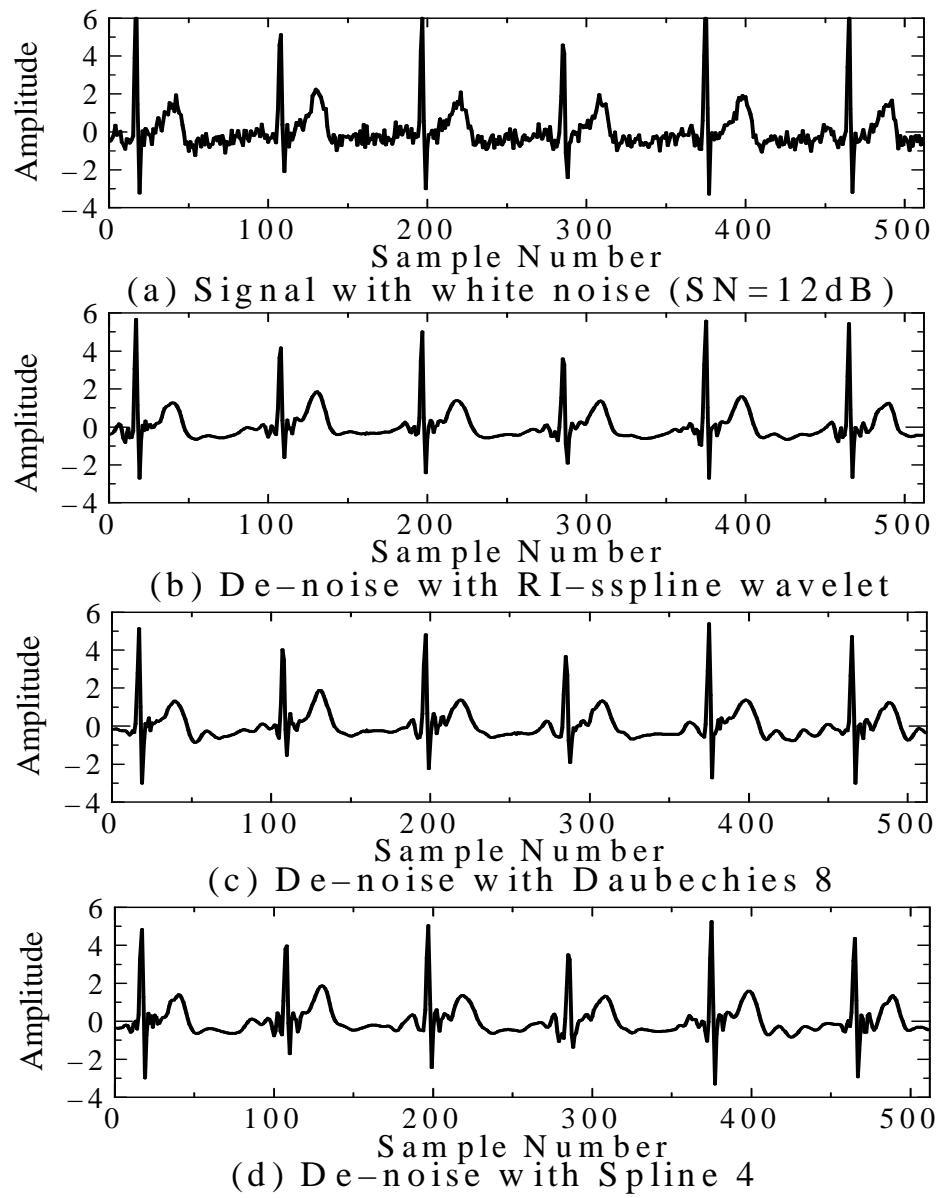
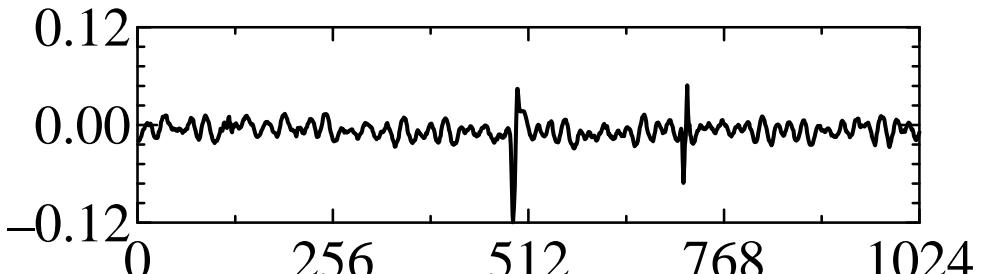
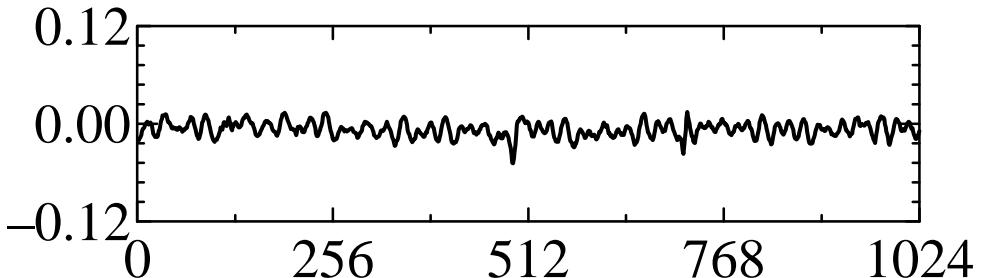


Fig.28 Results of de-noise (SN=12dB)

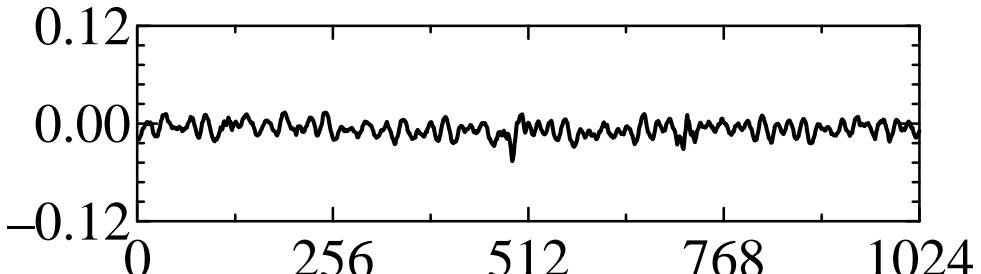
4.5.3 Music de-noise



(a) Music signal with pulse noise



(b) Noise removed by RI-Spline wavelet



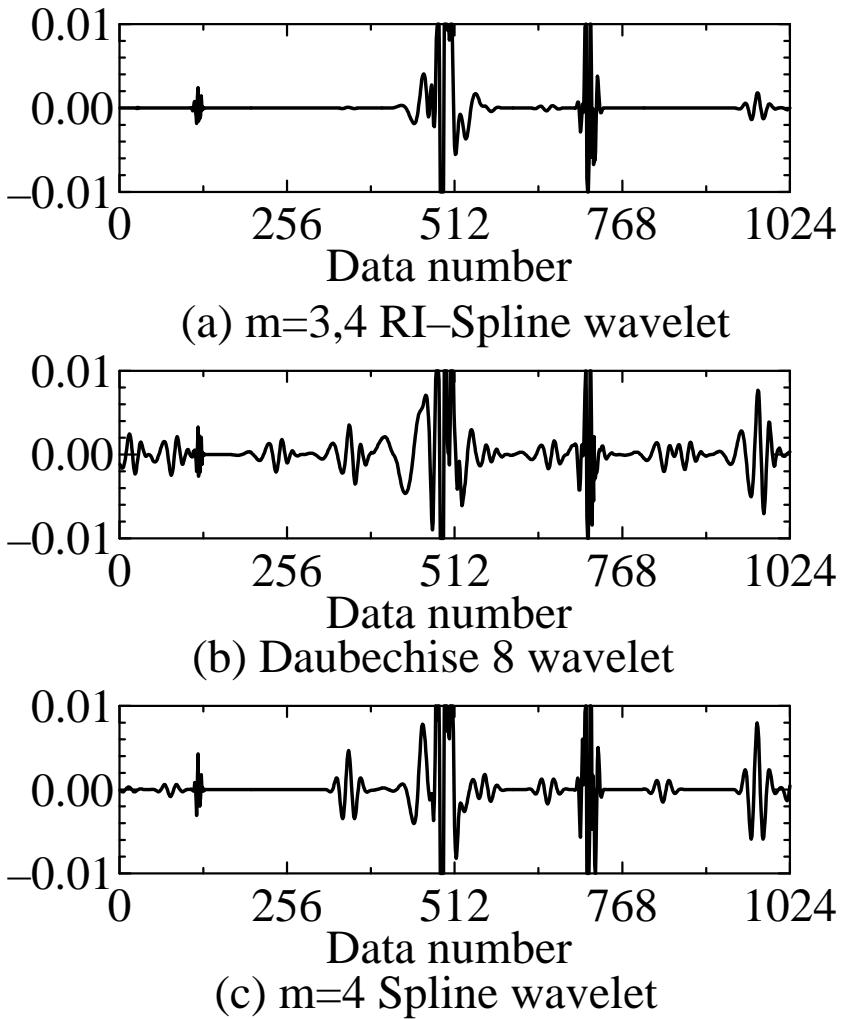
(c) Noise removed by Spline 4 wavelet

- (a) shows a music signal with impulse noise
- (b) and (c) show the signal after de-noise

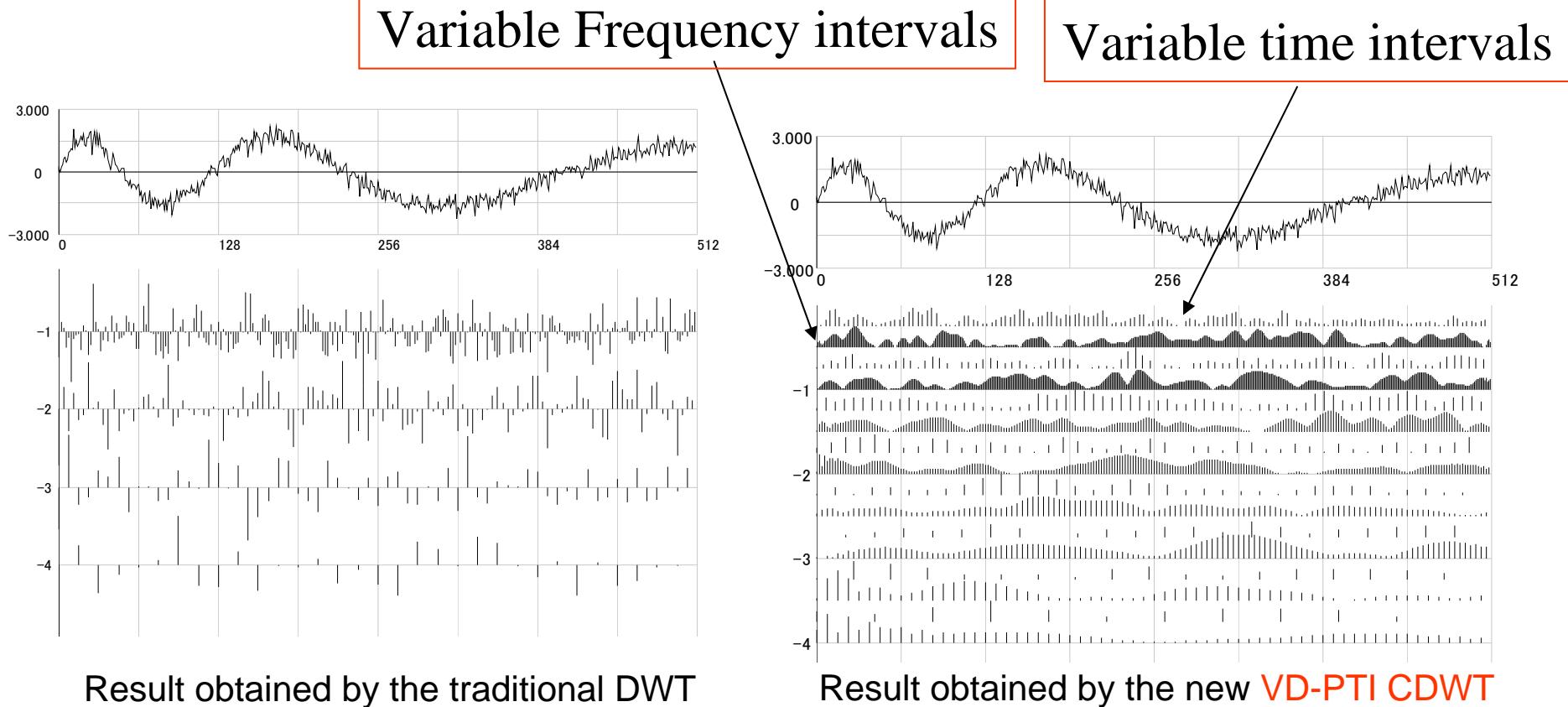
Results of de-noise



Figure shows impulse noise cut off. As shown in (b) and (c), by using D8 and Sp4 wavelet, the music signal also are cut off. Comparing to it, (a) by using RI-Spline Wavelet, the music signal almost is not cut off.

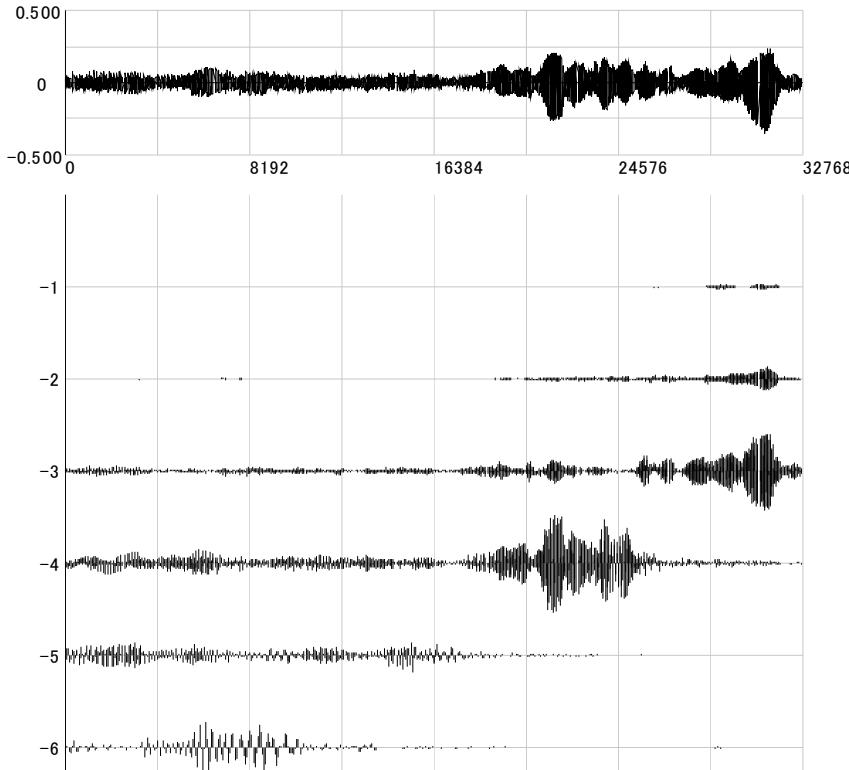


New research: Variable-density CDWT

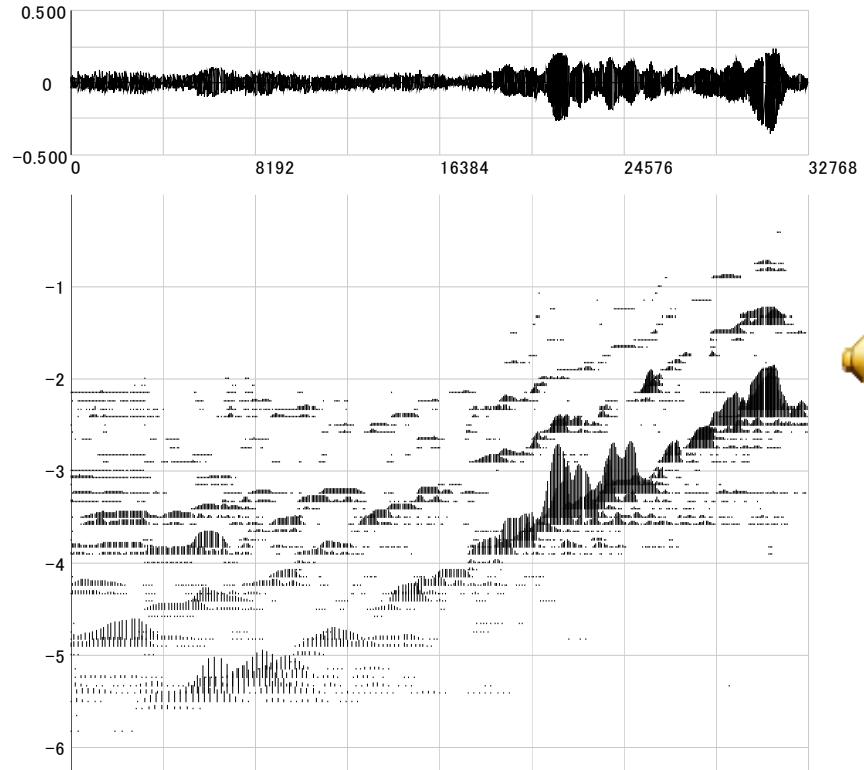


This VD transform can divide an octave frequency band into an arbitrary number of filter banks, and its wavelets can be arrayed on the time base with variable intervals.

Music analysis by using VD-PTI CDWT



Result obtained by DWTの



Result obtained by VD-PTI CDWT



VD-PTI CDWT by the analysis can shows sound of the invisible music but conventional DWT can not. In addition, 2 harmonic overtones of the musical instrument, 3 harmonic overtones precisely can be observed, too.

4. Discrete Wavelet Transform

4.1 Discrete Wavelet Transform (DWT)

4.2 Features of the DWT

Severe shift dependence.

Poor directional selectivity

4.3 New Design Method for CDWT

Make Real and Imaginary components of the complex

Mother Wavelet must be a Hilbert pair

4.4 New Calculation methods for CDWT

For scaling functions, which shapes of the real and imaginary components are completely same and positions are different from 1/2 sample

4.5 Example of De-noising

Model signals, EEG, Music